Rabinowitz-Floer homology on Brieskorn manifolds

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Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation selbsständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertig habe.

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I dedicate to her this thesis.

Contents

"... as simple and still as mysterious and complicated as a simple mathematical formula that can mean all happiness or can mean the end of the world."

Ernest Hemingway, *A moveable feast*

1. Introduction

1.1. Motivation

Contact geometry and topology (as well as its symplectic counterpart) is a relative new mathematical field. It dates essentially back to V. Arnold's seminal book "Mathematical Methods of Classical Mechanics",[3], from 1974 based on lectures he gave in the sixties. However, the basic structures and questions of the area arise from Newtonian/Hamiltonian mechanics. It is thus closely related to questions on the behaviour of mechanical systems, such as the whole universe. As an example, let us consider the motion of several planets: The total energy of a system of n points x_j of masses $m_1, \dots, m_n$ (e.g. a system of n planets) in $\mathbb{R}^3$ is given by

$$H := \sum_{j=1}^{n} \frac{1}{2}||\dot{x}_j||^2 - \sum_{1\leq j<k\leq n} \frac{m_j m_k}{||x_j - x_k||}.$$

Considering the phase space $V := \mathbb{R}^{3n} \times \mathbb{R}^{3n}$ and writing $q_j := x_j$ and $p_j := \dot{x}_j$, we find that the time evolution of this system is described by

$$\dot{x}_j = \dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \text{and} \qquad \ddot{x}_j = \dot{p}_j = -\frac{\partial H}{\partial q_j} = m_j \sum_{k\neq j} m_k \frac{x_k - x_j}{||x_k - x_j||^3},$$

where the second equation is Newton's law of gravity. We cannot solve this equation explicitly (not even for only 3 mass-points, a setup which is known as the 3-body problem). However, it is still possible to show that the system of solutions has certain properties. As is well-known, the total energy of the system is time-independent. This implies that every solution of the above differential equation stays, for all time, in a fixed energy hypersurface $\Sigma := H^{-1}(E_0) \subset V$. A puzzling question is to what extend can Σ tell us something about the dynamical behaviour of the solutions of the equation?
At first glance, the prospect of answering this seems hopeless. However, there are many classical examples which relate the topology of a space with the behaviour of a dynamical systems on this space. For example, a theorem by Hopf tells us that on S^2 every vector field has at least one zero and hence every ordinary differential equation on S^2 has at least one constant solution. It is clear that it should be possible to obtain more information if one takes into account not only Σ but more geometric features.
One possible additional structure in Hamiltonian mechanics is the contact structure ξ on Σ (see Section 1.2 for a precise definition) and indeed recent theorems in symplectic geometry show more properties of dynamical systems in the presence of ξ. A natural

question to ask is in how far Σ determines ξ, or stated differently: Can the additional information be extracted from Σ alone? The purpose of this thesis is to show that Σ does not uniquely determine ξ. In fact, in every dimension greater then 3, we show the existence of various Σ which admit infinitely many (fillable) contact structures ξ.
Another question asks to what extend Σ determines the topological/differentiable structure of the whole phase space V. As the latter comes with a symplectic structure ω which induces the contact structure on Σ, we could also ask if (Σ, ξ) determines (V, ω). The thesis gives a partial answer: The Main Theorem tells us that often we have that either Σ does not determine ξ or (Σ, ξ) does not determine (V, ω).
Our main tool to achieve these results is Rabinowitz-Floer homology, a variant of symplectic homology which can be thought of as a Morse homology on the (infinite dimensional) loop space of V. It is a machinery to obtain information on the structure of a space with the help of the solutions to a gradient like partial differential equation (see equation (3) in Section 1.5).
The following sections of the introduction first provide precise definitions of the objects that we use in this thesis. Then, we give a sketch of the construction of Rabinowitz-Floer homology before finally presenting our main results and the structure of the text.
We invite the reader to consult Appendix D for any questions concerning the setup, assumptions, sign and grading conventions used in this thesis.

1.2. Preliminaries

A ***symplectic manifold*** (V, ω) is a smooth $2n$-dimensional manifold V together with a non-degenerate, closed 2-form ω. This means that $d\omega = 0$ and that

$$\omega_p^* : T_pV \to T_p^*V, \quad X \mapsto \omega_p(X, \cdot)$$

is an isomorphism for all $p \in V$. Equivalently, we may require that $d\omega = 0$ and ω^n is a volume form. Such an ω is then called a ***symplectic structure*** on V. One calls ω ***exact***, if there exists a 1-form λ, such that $d\lambda = \omega$. If the context is clear, we will also write (V, λ) to denote an exact symplectic manifold.
A function $H \in C^\infty(V)$ on a symplectic manifold (V, ω) is called a ***Hamiltonian***. We define its ***Hamiltonian vector field*** X_H via

$$dH = -\iota(X_H)\omega = -\omega(X_H, \cdot) = \omega(\cdot, X_H).$$

A ***contact manifold*** Σ is a smooth $(2n-1)$-dimensional manifold together with a completely non-integrable smooth hyperplane distribution $\xi \subset T\Sigma$. The distribution is called a ***contact structure***. It is locally defined as $\xi = \ker \alpha$, where α is a local 1-form satisfying $\alpha \wedge (d\alpha)^{n-1} \neq 0$ pointwise. If α is globally defined (which we will always assume), it is called a ***contact form***. A global α gives rise to a volume form and Σ is then orientable. Once an orientation chosen, we will require that $\alpha \wedge (d\alpha)^{n-1} > 0$.
The ***Reeb vector field*** R_α of α is the unique vector field satisfying

$$\iota(R_\alpha)d\alpha = 0 \qquad \text{and} \qquad \iota(R_\alpha)\alpha = 1.$$

R_α is transverse to ξ and we have therefore $T\Sigma = \xi \oplus \mathbb{R}R_\alpha$. However, this splitting depends on the chosen contact form α, as R_α depends on α. Reeb trajectories of (Σ, α) are the trajectories of the flow of R_α, i.e. solutions $v \in C^\infty(\mathbb{R}, \Sigma)$ of the equation

$$\partial_t v(t) - R_\alpha(v(t)) = 0. \tag{1}$$

Reeb orbits are the images of Reeb trajectories.

Discussion 1. For two contact forms α and α' which define the same contact structure ξ, i.e. $\ker\alpha = \xi = \ker\alpha'$, we find a function f such that $\alpha' = f \cdot \alpha$. In fact, f is given by $f := \alpha'(R_\alpha)$ and has therefore no zeros. We will henceforth always assume that $\alpha' = f \cdot \alpha$ with $f > 0$. This is a minor restriction, since changing f to $-f$ replaces $R_{f\alpha}$ by $R_{-f\alpha} = -R_{f\alpha}$. This leaves the Reeb dynamics basically unchanged – the same Reeb orbits are run through with the same speed but in the opposite direction.

An ***almost complex structure*** J is a bundle-endomorphism of ξ or TV, such that $J^2 = -Id$. It is called α-compatible if $d\alpha(\cdot\,, J\cdot)$ defines a Riemannian metric on ξ. Similarly, one defines ω-compatible J. The space of α- resp. ω-compatible almost complex structures is non-empty and contractible.
Any pair (α, J) of a contact form α and an α-compatible almost complex structure J induces a reduction of the structure group of the tangent bundle $T\Sigma$ to the unitary group $1 \times U(n-1)$. This reduction is called an ***almost contact structure***. The ***formal homotopy class*** $[\xi]$ of a contact structure $\xi = \ker\alpha$ is the homotopy class of its almost contact structure. It does not depend on the particular choice of (α, J) and is hence well-defined.

Examples.

- $\mathbb{R}^{2n}$ with the 2-form $\omega_{std} := \sum_{k=1}^n dx_k \wedge dy_k$ is the standard model for a symplectic manifold. A ω_{std}-compatible almost complex structure J is given by $J\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)$. The unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ is a contact manifold with contact form α given by the restriction of the 1-form $\lambda := \frac{1}{2}\sum_{k=1}^n (x_k dy_k - y_k dx_k)$ to S^{2n-1}. The standard contact structure is $\xi_{std} := \ker\alpha$. The Reeb vector field is $R := 2(-y_1, x_1, \dots, -y_n, x_n)$.

- A different contact manifold is $\mathbb{R}^{2n+1}$ with contact form $\alpha = dz + \sum_{k=1}^n x_k dy_k$ and Reeb vector field $R = \partial_z$. A α-compatible almost complex structure is again given by $J\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)$.

- More general examples are obtained as follows. Let N be a smooth n-dimensional manifold and let T^*N be its cotangent bundle. Let q_k be local coordinates on N and let p_k be the associated cotangent coordinates, i.e. if $p \in T_q^*N$, then $p = \sum_{k=1}^n p_k \cdot dq_k$. Define locally a 1-form α and a 2-form ω on T^*N by

$$\alpha := -\sum_{k=1}^n p_k dq_k \qquad \text{and} \qquad \omega := \sum_{k=1}^n dq_k \wedge dp_k.$$

  Note that α is a primitive of ω. Both definitions are coordinate-independent and hence ω gives a symplectic structure on T^*N.

Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on N. This provides a scalar product on T_q^*N for every q. The unit cotangent bundle S^*N is the submanifold of T^*N which consists of points (q, p) with $\langle p, p \rangle_q = 1$. The 1-form α is a contact form on S^*N, as the Liouville vector field

$$Y_\alpha := \sum_{k=1}^{n} p_k \cdot \partial_{p_k}$$

is transverse to S^*N (see Lemma 3). The Reeb flow on S^*N is the geodesic flow with respect to $\langle \cdot, \cdot \rangle$.

1.3. Exact fillings for contact manifolds

There are two important constructions which link contact and symplectic manifolds:

- for us, the ***symplectization*** of a contact manifold (Σ, α) is $\Sigma \times \mathbb{R}$ endowed with the exact symplectic form $\omega = d(e^r \alpha)$, where r is a coordinate on $\mathbb{R}$;

- an ***exact contact hypersurface*** is a hypersurface $\Sigma \subset V$ of an exact symplectic manifold (V, λ), where the pull-back $\alpha := i^*\lambda$ by the inclusion gives a contact form on Σ.

Definition 2. *Let (V, λ) be an exact symplectic manifold. The unique vector field Y_λ which satisfies $\iota(Y_\lambda)\omega = \lambda$ is the **Liouville vector field** of λ.*

Lemma 3. *A hypersurface $\Sigma \subset V$ is an exact contact hypersurface of (V, λ) if and only if Y_λ is transverse to $T\Sigma$ along Σ.*

Proof: The form $\alpha \wedge (d\alpha)^{n-1}$ is non-degenerate if and only if $Y_\lambda \notin T\Sigma$, since

$$\alpha \wedge (d\alpha)^{n-1} = i^*\lambda \wedge (di^*\lambda)^{n-1} = i^*\big(\lambda \wedge (d\lambda)^{n-1}\big) = i^*\big(\iota(Y_\lambda)\omega \wedge \omega^{n-1}\big) = i^*\big(\iota(Y_\lambda)\omega^n\big). \ \square$$

Definition 4. *A **Liouville domain** is a compact exact symplectic manifold (V, λ) with boundary Σ, such that the Liouville vector field Y_λ points outwards along Σ.*

It follows from Lemma 3 that the boundary Σ of a Liouville domain (V, λ) is an exact contact hypersurface. Moreover, the proof of Lemma 3 shows that the orientation of Σ as the boundary of V coincides with the orientation given by $\alpha \wedge (d\alpha)^{n-1}$.

Discussion 5. We make the following observations for a Liouville vector field Y_λ, using Cartan's magic formula:

$$\mathcal{L}_{Y_\lambda}\omega = \iota(Y_\lambda)d\omega + d(\iota(Y_\lambda)\omega) = d\lambda = \omega$$
$$\mathcal{L}_{Y_\lambda}\lambda = \iota(Y_\lambda)d\lambda + d(\iota(Y_\lambda)\lambda) = \iota(Y_\lambda)\omega = \lambda$$

since $d\omega = 0$ and $\iota(Y_\lambda)\lambda = \omega(Y_\lambda, Y_\lambda) = 0$. Hence, the flow ϕ of Y_λ satisfies

$$(\phi^r)^*\lambda = e^r \cdot \lambda \qquad \text{and} \qquad (\phi^r)^*\omega = e^r \cdot \omega.$$

If we consider a Liouville domain (V, λ), we find that the negative half flow ϕ^r, $r \in (-\infty, 0]$, of Y_λ is complete as V is compact and Y_λ points outwards along $\partial V = \Sigma$, so that ϕ can never leave V in negative time. Combining these two facts, we can find in any Liouville domain a collar neighborhood of Σ which is symplectomorphic via ϕ to $(\Sigma \times (-\infty, 0], e^r \cdot \alpha)$, the non-positive symplectization of (Σ, α).

Definition 6. *The* ***completion*** $(\hat{V}, \hat{\lambda})$ *of a Liouville domain* (V, λ) *is obtained by glueing the positive symplectization of* $\Sigma = \partial V$ *to* V *along* Σ*. In other words, it is the exact symplectic manifold*

$$\hat{V} := V \cup_\phi (\Sigma \times \mathbb{R}), \qquad \hat{\lambda} := \begin{cases} \lambda & \text{on } V \\ e^r \cdot \alpha & \text{on } \Sigma \times \mathbb{R} \end{cases}$$

where we identify $\Sigma \times (-\infty, 0]$ *with the collar of* Σ *in* V *as described above.*

Examples.

- In a symplectization $(\Sigma \times \mathbb{R}, e^r \alpha)$, the Liouville vector field is ∂_r – the partial derivative with respect to the coordinate on $\mathbb{R}$ – and its flow ϕ is simply given by
$$\phi^t(x, r) = (x, r + t) \qquad \forall r, t \in \mathbb{R}.$$
- The unit ball in $(\mathbb{R}^{2n}, \omega_{std})$ is a Liouville domain with contact boundary (S^{2n-1}, α). The Liouville vector field is $Y_\lambda = \frac{1}{2}(x_1, y_1, \dots, x_2, y_2)$. Moreover, $(\mathbb{R}^{2n}, \omega_{std})$ is the completion of the unit ball.
- More generally, the unit disk bundle D^*N in T^*N (given by pairs (q, p) with $\langle p, p\rangle_q \leq 1$) is a Liouville domain with contact boundary S^*N and Liouville vector field Y_α. Again, T^*N is itself the completion of D^*N.

Definition 7. *A* ***Liouville isomorphism*** *between Liouville domains* (V_1, λ_1), (V_2, λ_2) *is a diffeomorphism* $\varphi : \hat{V}_1 \to \hat{V}_2$ *satisfying* $\varphi^* \hat{\lambda}_2 = \hat{\lambda}_1 + dg$ *for a compactly supported function* g*.*

Proposition 8 ([48], page 3)**.** *For any Liouville isomorphism* $\varphi : \hat{V}_1 \to \hat{V}_2$ *there exists an* $R > 0$ *such that on* $\Sigma_1 \times [R, \infty) \subset \hat{V}_1$ *the map* φ *has the following form:*

$$\varphi(r, x) = (\psi(x), r - f(x)),$$

where $\psi : \partial V_1 = \Sigma_1 \to \Sigma_2 = \partial V_2$ *is a contact isomorphism satisfying* $\psi^* \alpha_2 = e^f \cdot \alpha_1$ *for a function* $f \in C^\infty(\Sigma_1)$*. So near* ∞*, the map* ϕ *is essentially a coordinate change in* r*.*

Proof: As $\varphi^* \hat{\lambda}_2 = \hat{\lambda}_1 + dg$ with $supp(g)$ compact, we can find an R such that $\varphi^* \hat{\lambda}_2 = \hat{\lambda}_1$ on $\Sigma_1 \times [R, \infty)$. This implies $\hat{\omega}_1 = d\hat{\lambda}_1 = d\varphi^* \hat{\lambda}_2 = \varphi^*(d\hat{\lambda}_2) = \varphi^* \hat{\omega}_2$ and hence also $\varphi_* Y_{\hat{\lambda}_1} = Y_{\hat{\lambda}_2}$. On $\Sigma_1 \times [R, \infty)$, φ is therefore compatible with the flows of $Y_{\hat{\lambda}_1}$ resp. $Y_{\hat{\lambda}_2}$ in the sense that $\phi^t_{\hat{\lambda}_2} \circ \varphi = \varphi \circ \phi^t_{\hat{\lambda}_1}$. This implies that for every $y \in \Sigma_2$ the flow line $\{y\} \times \mathbb{R} \subset \hat{V}_2$ with respect to $\phi_{\hat{\lambda}_2}$ is hit at most once by $\varphi(\Sigma_1 \times \{R\})$, as φ is injective.

Since $\varphi(\hat{V}_1 \setminus (R, \infty)) \subset \hat{V}_2$ is compact, we know that every $\{y\} \times \mathbb{R}$ intersects $\varphi(\Sigma_1 \times [R, \infty))$ and by following the flow of $\phi_{\hat{\lambda}_2}$ backwards, we know that $\{y\} \times \mathbb{R}$ even intersects $\varphi(\Sigma_1 \times \{R\})$. Therefore, $\varphi(\Sigma \times \{R\}) \cap \big(\{y\} \times \mathbb{R}\big)$ contains exactly one element and we may write

$$\varphi(x, R) = (\psi(x), \tilde{f}(x)),$$

where $\psi : \Sigma_1 \to \Sigma_2$ is a diffeomorphism and $\tilde{f} \in C^\infty(\Sigma_1)$. The compatibility of φ with the two flows shows that

$$\varphi(x, r) = (\psi(x), \tilde{f}(x) + r - R) \qquad \forall r \geq R.$$

Note that the 1-form λ_2 at $\psi(x, \tilde{f}(x))$ is given by $e^{\tilde{f}(x)} \cdot \alpha_2$. Since $\varphi^* \hat{\lambda}_2 = \hat{\lambda}_1$ on $\Sigma_1 \times [R, \infty)$, we find that

$$\psi^* \left(e^{\tilde{f}(x)} \cdot \alpha_2 \right) = e^R \cdot \alpha_1.$$

Setting $f := R - \tilde{f}$, we then have $\psi^* \alpha_2 = e^f \cdot \alpha_1$ and $\varphi(x, r) = (\psi(x), r - f(x))$. $\square$

Note that, while Liouville isomorphisms preserve the contact structure of the boundary, the contact form may change arbitrarily. More precisely: Consider a Liouville domain (V, λ) with contact boundary (Σ, α). If $\alpha' = e^f \cdot \alpha$ is another contact form which defines the same contact structure, we may consider the following contact hypersurface in the completion $\hat{V}$:

$$\Sigma' = \{(x, f(x)) \,|\, x \in \Sigma\}.$$

Obviously, Σ' bounds a compact region $V' \subset \hat{V}$, so that $(V', \lambda' := \hat{\lambda}|_{V'})$ is a Liouville domain, whose completion is also $(\hat{V}, \hat{\lambda})$. Hence, (V, λ) and (V', λ') are Liouville isomorphic with trivial diffeomorphism φ. This motivates the following definition:

Definition 9. *Let (Σ, ξ) be a contact manifold. If there exists a Liouville domain (V, λ) such that $\partial V = \Sigma$ and $\xi = \ker i^* \lambda$, then we call the equivalence class of (V, λ) under Liouville isomorphisms an* ***exact contact filling*** *of (Σ, ξ).*

The discussion above shows that an exact contact filling does not depend on a specific contact form. Therefore, any invariant of exact contact fillings gives an invariant for contact structures (with the filling). We will later show (Corollary 56), that each exact contact filling of (Σ, ξ) possesses a well-defined Rabinowitz-Floer homology, which is therefore an invariant of the contact structure (together with the filling).

1.4. Defining Hamiltonians

The setup in which we define Rabinowitz-Floer homology is the following. Let (V, λ) be the completion of a Liouville domain $\tilde{V}$ with contact boundary $M := \partial \tilde{V}$. Let $\Sigma \subset V$ be an exact contact hypersurface bounding a compact domain $W \subset V$, so that Σ is the boundary of the Liouville domain W. In particular, we do not require that the completions $\widehat{W}$ and $\widehat{\tilde{V}} = V$ coincide (nevertheless, $\widehat{W} \subset V$ as symplectic submanifold).

Definition 10. *A **defining Hamiltonian** for the boundary Σ of a Liouville domain $W \subset V$ is a function $H \in C^\infty(V)$, which is constant outside a compact set, whose zero level set $H^{-1}(0)$ equals Σ and whose Hamiltonian vector field X_H agrees with the Reeb vector field R_α on Σ, i.e. for the inclusion $i : \Sigma \hookrightarrow V$ holds*

$$\alpha := i^*\lambda \qquad \textit{and} \qquad i_*(R_\alpha) = X_H|_\Sigma.$$

In particular, note that Σ is a regular level set of H.

Examples. Let β be a smooth monotone cut-off function such that $\beta(x) = \begin{cases} x & x \leq 2 \\ 3 & x \geq 4 \end{cases}$.

- The function $H(p) := \beta\big(||p||^2\big) - 1$ is a defining Hamiltonian for S^{2n-1} in $(\mathbb{R}^{2n}, \omega_{std})$.
- The function $H(q,p) := \beta\big(\langle p,p\rangle_q\big) - 1$ is a defining Hamiltonian for S^*N in T^*N.

Proposition 11. *For the boundary Σ of a Liouville domain $W \subset V$ holds:*

i. The space $\mathscr{H}$ of defining Hamiltonians for Σ is non-empty and convex.

ii. If α_0 and α_1 are two contact forms defining the same contact structure on Σ, then there exists a homotopy of Liouville domains $(W_s, \Sigma_s) \subset (V, \lambda)$ and a corresponding homotopy of defining Hamiltonians H_s, such that $\alpha_0 = \lambda|_{\Sigma_0}$ and $\alpha_1 = \lambda|_{\Sigma_1}$.

Proof: Since (V, λ) is the completion of a Liouville domain, we find that the Liouville vector field Y_λ is complete. This allows us to find a symplectic embedding of the symplectization $i : (\Sigma \times \mathbb{R}) \to V$ with $i(\Sigma \times \{0\}) = \Sigma$ (see Discussion 5). Hence, it suffices to construct defining Hamiltonians on $\Sigma \times \mathbb{R}$.

@i. Here, we consider the function $\qquad \tilde{H}(x,r) := e^{\rho(r)} - 1,$

where ρ is a smooth monotone increasing function with $\rho(r) = r$ near 0 and ρ constant outside a compact set. This guarantees that $\tilde{H}^{-1}(0) = \{r = 0\} = \Sigma \times \{0\}$. Since $d\tilde{H} = e^r dr$ near $\Sigma \times \{0\}$ and $\iota_{R_\alpha}\omega = \iota_{R_\alpha}\big(d(e^r\alpha)\big) = \iota_{R_\alpha}\big(e^r d\alpha + e^r dr \wedge \alpha\big) = -e^r dr$, the Hamiltonian vector field agrees with R_α on $\Sigma \times \{0\}$.
The defining Hamiltonian H for Σ is then obtained from $\tilde{H} \circ i^{-1}$ by extending it as constant on $V \setminus i(\Sigma \times \mathbb{R})$. Hence $\mathscr{H} \neq \varnothing$. In general, H is a defining Hamiltonian for Σ if and only if it satisfies the following equation:

$$\mathcal{L}_{Y_\lambda} H|_\Sigma = dH(Y_\lambda)|_\Sigma = \omega(Y_\lambda, R_\alpha) = \lambda(R_\alpha) = 1.$$

Since $H(\Sigma) = 0$, this implies that every defining H is positive on $\Sigma \times \mathbb{R}^+$ and negative on $\Sigma \times \mathbb{R}^-$. Hence we have for $H_1, H_2 \in \mathscr{H}$ that

$$s \cdot H_1 + (1-s) \cdot H_2 \in \mathscr{H} \qquad \text{for all} \quad s \in [0,1].$$

@ii. Without loss of generality, we may assume that $\alpha_0 = i^*\lambda$, where $i : \Sigma \times \{0\} \hookrightarrow V$ is the embedding mentioned above. Since α_0 and α_1 define the same contact structure on Σ, we find a function $f \in C^\infty(\Sigma)$, such that $\alpha_1 = e^f \cdot \alpha_0$. Then we modify the above construction as follows:

$$\tilde{H}_s(x, r) := e^{\rho(r - s\cdot f(x))} - 1.$$

Define H_s again by extending $\tilde{H}_s \circ i^{-1}$ as constant on $V \setminus i(\Sigma \times \mathbb{R})$. The homotopy of Liouville domains is then given by $W_s := H_s^{-1}\big((\infty, 0]\big)$ and $\Sigma_s := H_s^{-1}(0)$. $\square$

1.5. Rabinowitz-Floer homology

As above, let (V, λ) be the completion of a Liouville domain $\tilde{V}$ with contact boundary $M = \partial\tilde{V}$, let $\Sigma \subset V$ be an exact contact hypersurface bounding a compact Liouville domain W and let H be a defining Hamiltonian for Σ.

Definition 12. *The **spectrum** $spec(\Sigma, \alpha)$ of a contact manifold Σ with contact form α is the set of real numbers $\eta \in \mathbb{R}$ such that the ordinary differential equation $\dot{v} = \eta R_\alpha$ has a 1-periodic solution. We denote with $\mathcal{P}(\alpha) \subset C^\infty(S^1, \Sigma) \times \mathbb{R}$ the set of pairs (v, η), such that $\eta \in spec(\Sigma, \alpha)$ and $\dot{v} = \eta R_\alpha$.*

Remark. We identify a pair $(v, \eta) \in \mathcal{P}(\alpha)$ with the η-periodic Reeb orbit $\tilde{v}(t) := v(t/\eta)$. Note that we do not exclude the cases $\eta = 0$ and $\eta < 0$. For $\eta < 0$, we have that v is again a Reeb orbit with period $|\eta|$, but run through in the opposite direction. For $\eta = 0$, the loop v is just constant. The pairs $(v, 0) \in \mathcal{P}(\alpha)$ are hence in one-to-one correspondence to the points of Σ.

Let $\mathscr{L} = C^\infty(S^1, V)$ denote the free (smooth) loop space of V. The ***Rabinowitz action functional*** on V associated to H is given by

$$\mathcal{A}^H : \mathscr{L} \times \mathbb{R} \to \mathbb{R}, \qquad \mathcal{A}^H(v, \eta) := \int_0^1 \Big(\lambda(\dot{v}(t)) - \eta H(v(t))\Big) dt.$$

One can think of $\mathcal{A}^H$ as the Lagrange multiplier action functional of classical mechanics. In Section 2.1, we will see that the critical points (v, η) of $\mathcal{A}^H$ are characterized by

$$0 = \dot{v} - \eta X_H \qquad \text{and} \qquad 0 = \int_0^1 H(v(t))dt.$$

The first equation implies that $\frac{d}{dt}H(v) = dH(\dot{v}) = dH(\eta X_H) = 0$, so that $H(v)$ is constant and hence 0 by the second equation. As H is a defining Hamiltonian with $H^{-1}(0) = \Sigma$ and $X_H|_\Sigma = R_\alpha$, we find that these equations are therefore equivalent to

$$v(t) \in \Sigma \quad \forall\, t \qquad \text{and} \qquad \dot{v} = \eta R_\alpha. \tag{2}$$

This shows that the set of critical points $crit\big(\mathcal{A}^H\big)$ of $\mathcal{A}^H$ consists of the closed Reeb trajectories v on Σ with period η, or equivalently that $crit\big(\mathcal{A}^H\big) = \mathcal{P}(\alpha)$.

Definition 13. *A family of ω-compatible almost complex structures*

$$J : S^1 \times \mathbb{R} \to End(TV), \quad (t,n) \mapsto J_t(\cdot, n)$$

is called ***admissible*** *(of class C^ℓ/smooth) if:*

- *as a map with domain $S^1 \times V \times \mathbb{R}$ we have that J is of class C^ℓ/smooth.*
- *J is t- and n-independent cylindrical at the unbounded end in V. This means that on $M \times [R, \infty)$ for $R > 0$ sufficiently large J is of the form*

$$J|_{\xi_M} = J_0 \qquad \text{and} \qquad J\frac{\partial}{\partial r} = R_\lambda,$$

where J_0 is any compatible almost complex structure on the contact structure ξ_M of M and R_λ is the vector field whose restriction to the contact hypersurface $M \times \{r\}$ equals the Reeb vector field. Equivalently, we can require that $d(e^r) \circ J = -\lambda$ on $M \times [R, \infty)$.

- *the family is C^ℓ-bounded, meaning that $\sup_n ||J_t(\cdot, n)||_{C^\ell} < \infty$ with respect to some background norm $|| \cdot ||$ on M.*

Remark. The reason for the dependency of J on the additional parameter n, not found in the literature until recently in [8] and [1], is based in the transversality problem for Rabinowitz-Floer homology and will become clear in the proof of the local Transversality Theorem 38.

In Section 2.1, formula (6), we will see that such a family of almost complex structures J can be used to define a metric g on $\mathscr{L} \times \mathbb{R}$, which yields a gradient $\nabla \mathcal{A}^H$ for $\mathcal{A}^H$. The explicit formula for $\nabla \mathcal{A}^H$ is given in (7).

Definition 14. *An* $\mathcal{A}^{\mathbf{H}}$***-gradient trajectory*** *is a solution $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$ of the Rabinowitz-Floer equation, which is the following partial differential equation:*

$$\partial_s(v,\eta) = \nabla \mathcal{A}^H(v,\eta) \qquad \Leftrightarrow \qquad \begin{aligned} \partial_s v + J_t(v,\eta)\big(\partial_t v - \eta X_H(v)\big) &= 0 \\ \partial_s \eta \; + \int_0^1 H\big(v(s,t)\big)dt &= 0. \end{aligned} \tag{3}$$

The next two lemmas characterize the $\mathcal{A}^H$-gradient trajectories which connect critical points $(v^\pm, \eta^\pm) \in crit\big(\mathcal{A}^H\big) = \mathcal{P}(\alpha)$.

Lemma 15. *If $(v,\eta) \in \mathcal{P}(\alpha)$, then $\mathcal{A}^H(v,\eta) = \eta$. Thus $\mathcal{A}^H(\mathcal{P}(\alpha)) = spec(\Sigma, \alpha)$.*

Proof: Since $H|_\Sigma = 0$ and $\dot v = \eta R$ and $im(v) \subset \Sigma$ if $(v,\eta) \in \mathcal{P}(\alpha)$, we may calculate

$$\mathcal{A}^H(v,\eta) = \int_0^1 \Big(\lambda(\dot v) + \eta H(v)\Big)dt = \int_0^1 \lambda(\eta R)dt = \int_0^1 \eta \, dt = \eta. \qquad \square$$

Lemma 16. *If (v, η) is a non-stationary $\mathcal{A}^H$-gradient trajectory between critical points $(v^\pm, \eta^\pm) \in \mathcal{P}(\alpha)$, i.e. where $\lim_{s \to \pm\infty} (v, \eta) = (v^\pm, \eta^\pm)$, then $\eta^+ > \eta^-$.*

Proof: With $|| \cdot ||$ as the norm of the metric g and Lemma 15, we calculate

$$\begin{aligned} \eta_+ - \eta_- = \mathcal{A}^H(v^+, \eta^+) - \mathcal{A}^H(v^-, \eta^-) &= \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}^H(v, \eta) ds \\ &= \int_{-\infty}^{\infty} g\big(\nabla \mathcal{A}^H(v, \eta), \partial_s(v, \eta)\big) ds \\ &\overset{(3)}{=} \int_{-\infty}^{\infty} ||\nabla \mathcal{A}^H(v, \eta)||^2 ds \\ &> 0. \end{aligned}$$

□

Definition 17. *The quantity $E(v, \eta) := \int_{-\infty}^{\infty} ||\nabla \mathcal{A}^H(v, \eta)||^2 ds \geq 0$ is called the* ***energy*** *of the $\mathcal{A}^H$-gradient trajectory (v, η). If $\lim_{s \to \pm\infty} (v, \eta) = (v^\pm, \eta^\pm) \in \mathcal{P}(\alpha)$, then Lemma 16 tells us that $E(v, \eta) = \eta^+ - \eta^-$.*

Since any pair $(v, \eta) \in \mathcal{P}(\alpha)$, $\eta \neq 0$, yields a whole S^1-family of points in $\mathcal{P}(\alpha)$ by time shift, $\mathcal{P}(\alpha)$ never consists of isolated points. In order to build a Floer-type homology with basis $\mathcal{P}(\alpha)$, we therefore have to use Morse-Bott techniques. This is why we impose the following non-degeneracy assumption on the Reeb flow ϕ^t on Σ:

> *The set $\mathcal{N}^\eta \subset \Sigma$ formed by the η-periodic Reeb orbits is a closed submanifold for each $\eta \in \mathbb{R}$ and $T_p \mathcal{N}^\eta = \ker(D_p \phi^\eta - id)$ holds for all $p \in \mathcal{N}^\eta$.* (MB)

Note that we do not assume that the rank $d\lambda|_{\mathcal{N}^\eta}$ is locally constant, as was done in [14]. As far as we see this is not needed. The assumption (MB) is generically satisfied, as shown in [14], appendix B. Moreover, (MB) implies that $\mathcal{P}(\alpha)$ is a proper submanifold of $\mathscr{L} \times \mathbb{R}$ (see Theorem 23). Note that the components of $\mathcal{P}(\alpha)$ with fixed $\eta \in spec(\Sigma, \alpha)$ correspond to the submanifolds $\mathcal{N}^\eta$ of Σ via the map $(v, \eta) \mapsto v(0)$. By abuse of notation, we will write $\mathcal{N}^\eta$ also for the components of $\mathcal{P}(\alpha)$, i.e. we consider $\mathcal{N}^\eta$ either as a submanifold of $\mathscr{L} \times \mathbb{R}$ or as a submanifold of Σ, depending on the context.

There are several approaches to deal with Morse-Bott situations. The one that we will use here, *flows with cascades*, was developed by Urs Frauenfelder, [25], and Frédéric Bourgeois, [5], based on an idea from Piunikhin, Salamon, and Schwarz. It uses flows on the critical manifolds without further perturbation, which makes the computations we have in mind easier. For that, we choose an additional Morse function h and a suitable metric g_h on $\mathcal{P}(\alpha)$ such that the restrictions $h_\eta := h|_{\mathcal{N}^\eta}$ and $g_\eta := g_h|_{\mathcal{N}^\eta}$ form a Morse-Smale pair on $\mathcal{N}^\eta$ for every $\eta \in spec(\Sigma, \alpha)$. This is equivalent to h_η being a Morse function on $\mathcal{N}^\eta$ for which the stable and unstable manifolds $W^s(p)$ resp. $W^u(q)$ of the $\nabla_{g_h} h$-gradient flow intersect transversally for each pair $p, q \in crit(h_\eta)$.

Definition 18. *An* ***h-Morse flow line*** *$y \in C^\infty\big(\mathbb{R}, crit(\mathcal{A}^H)\big)$ is a solution of $\dot{y} = \nabla h(y)$, where ∇h is the gradient of h with respect to g_h.*

Definition 19. *For $c^-, c^+ \in crit(h)$ and $m \in \mathbb{N}$, we call a* ***trajectory from*** *c^-* ***to*** *c^+* ***with*** *m* ***cascades*** *a tupel*

$$(x, t) = \big((x_k)_{1 \le k \le m}, (t_k)_{1 \le k \le m-1}\big),$$

consisting of $\mathcal{A}^H$-gradient trajectories $x_k = (v_k, \eta_k)$ and real numbers $t_k \ge 0$ such that there exist (possibly constant) h-Morse flow lines y_k, $0 \le k \le m$, with

i. $\lim_{s \to -\infty} y_0(s) = c^-, \qquad \lim_{s \to -\infty} x_1(s) = y_0(0),$

ii. $\lim_{s \to \infty} x_k(s) = y_k(0), \qquad \lim_{s \to -\infty} x_{k+1}(s) = y_k(t_k),$

iii. $\lim_{s \to \infty} x_m(s) = y_m(0), \qquad \lim_{s \to \infty} y_m(s) = c^+.$

A ***trajectory with*** *0* ***cascades from*** *c^-* ***to*** *c^+ is an h-Morse flow line from c^- to c^+. See Figure 1 for an illustration.*

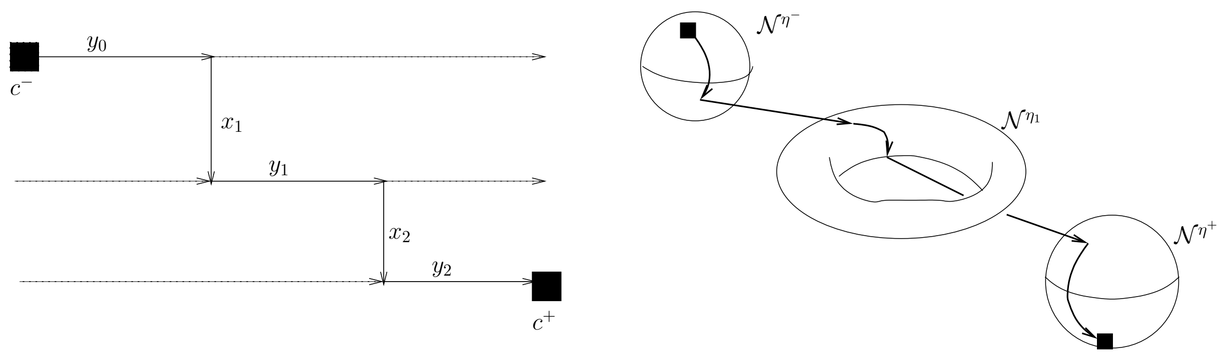

Fig. 1: A flow line with 2 cascades passing through the critical manifolds $\mathcal{N}^{\eta^-}$, $\mathcal{N}^{\eta_1}$, $\mathcal{N}^{\eta^+}$.

In other words, a trajectory with cascades is an alternating sequence of segments of h-Morse flow lines and whole $\mathcal{A}^H$-gradient trajectories. The space of all such trajectories with m cascades from c^- to c^+ is denoted by $\widehat{\mathcal{M}}(c^-, c^+, m)$. The moduli space

$$\mathcal{M}(c^-, c^+, m) := \widehat{\mathcal{M}}(c^-, c^+, m)\big/\mathbb{R}^m$$

is obtained by dividing out the free $\mathbb{R}^m$-action on the m cascades given by time shifts. If $m = 0$, we also divide by $\mathbb{R}$ (not by $\mathbb{R}^0 \cong 1$), as there is still an $\mathbb{R}$-action by time shift now on the h-Morse flow line. In Theorem 38 we show that $\mathcal{M}(c^-, c^+, m)$ is a manifold for generic choices of $J_t(\cdot, n)$. We denote the moduli space of all trajectories with cascades from c^- to c^+ by

$$\mathcal{M}(c^-, c^+) := \bigcup_{m \in \mathbb{N}} \mathcal{M}(c^-, c^+, m).$$

Due to Theorem 23, there are only finitely many non-empty critical manifolds $\mathcal{N}^\eta$ with $\eta \in [\eta^-, \eta^+]$ for each $\eta^\pm \in \mathbb{R}$, so that the above union is in fact finite. Indeed, each cascade reduces the period η (cf. Lemma 16) and connects critical manifolds $\mathcal{N}^{\eta_1}$ and $\mathcal{N}^{\eta_2}$, where $\eta^- \le \eta_1 < \eta_2 \le \eta^+$.

Theorem 46 states that $\mathcal{M}(c^-, c^+)$ still carries the structure of a manifold and Theorem 48 asserts that it is compact. Its zero-dimensional component $\mathcal{M}^0(c^-, c^+)$ is hence a finite set.

Now we define the Rabinowitz-Floer homology. The chain complex $RFC(H, h)$ is given as the $\mathbb{Z}_2$-vector space consisting of formal sums

$$\xi = \sum_{c \in crit(h)} \xi_c \cdot c,$$

where the coefficients $\xi_c \in \mathbb{Z}_2$ satisfy the following finiteness condition:

$$\#\{c \in crit(h) \,|\, \xi_c \neq 0 \,\wedge\, \mathcal{A}^H(c) \geq \kappa\} < \infty \qquad \text{for all } \kappa \in \mathbb{R}. \tag{4}$$

So $RFC(H, h)$ is a Novikov-completion of the $\mathbb{Z}_2$-vector space generated by the critical points of h. Let $\#_2\mathcal{M}^0(c^-, c^+) \in \mathbb{Z}_2$ denote the cardinality of $\mathcal{M}^0(c^-, c^+)$ modulo 2. The boundary operator ∂^F is then defined to be the (infinite) linear extension of

$$\partial^F c^+ = \sum_{c^- \in crit(h)} \#_2\mathcal{M}^0(c^-, c^+) \cdot c^-, \qquad c^+ \in crit(h). \tag{5}$$

To see that ∂^F is well-defined, we have to show that the right hand side still satisfies the finiteness condition (4):
Indeed, it follows from Lemma 16 that ∂^F reduces the action, i.e. $\#_2\mathcal{M}^0(c^-, c^+) \neq 0$ only if $\mathcal{A}^H(c^-) \leq \mathcal{A}^H(c^+)$. Moreover, we show in Theorem 23, that for any c^+ and every $a \in \mathbb{R}$, there are only finitely many $c^- \in crit(h)$ with $a \leq \mathcal{A}^H(c^-) \leq \mathcal{A}^H(c^+)$. Therefore $\partial^F c^+$ satisfies (4) for any c^+. In the general case, where $\partial^F\big(\sum \xi_c \cdot c\big) := \sum \xi_c \cdot \partial^F c$, condition (4) follows as both the sum $\sum \xi_c \cdot c$ and $\partial^F c$ satisfy the finiteness condition.

It follows from standard Floer-techniques by considering the 1-dimensional component of $\widehat{\mathcal{M}}(c^-, c^+)$, that $\partial^F \circ \partial^F = 0$. Hence, ∂^F is a boundary operator on $RFC(H, h)$. The ***Rabinowitz-Floer homology*** of (V, Σ) with respect to the Hamiltonian H and the Morse-function h is the homology of the chain complex $\big(RFC(H, h), \partial^F\big)$, i.e.

$$RFH(H, h) := \frac{\ker \partial^F}{\operatorname{im} \partial^F}.$$

We prove in Corollary 56 that $RFH(H, h)$ only depends on the contact manifold (Σ, ξ) and the filling Liouville domain W, thus allowing us to write $RFH(W, (\Sigma, \xi))$, where we omit ξ whenever it is clear from the context.
In Section 3.4, we show that $RFH(W, \Sigma)$ can be given a $\mathbb{Z}$-grading under the following assumptions:

(A) *The map $i_* : \pi_1(\Sigma) \to \pi_1(W)$ induced by the inclusion is injective.*

(B) *The integral $I_{c_1} : \pi_2(W) \to \mathbb{Z}$ of the first Chern class $c_1(TW)$ vanishes on spheres.*

For $c = (v, \eta) \in crit(h) \subset \mathcal{P}(\alpha)$, the degree $\mu(c)$ is a half integer given by the formula

$$\mu(c) = \mu_{CZ}(v) + ind_h(c) - \frac{1}{2} dim_c \mathcal{N}^\eta + \frac{1}{2},$$

where $ind_h(c)$ is the Morse index of c with respect to h, $\mu_{CZ}(v)$ is the (transversal) Conley-Zehnder index of v and $dim_c \mathcal{N}^\eta$ is the local dimension of $\mathcal{N}^\eta$ at c.

1.6. Outline of the thesis and main result

A major part of this thesis is devoted to technical details for the construction of Rabinowitz-Floer homology. We do this, since up to now, there is no complete reference for this in the literature. Note that due to the integral term in equation (3) some delicate adaptations to the standard construction by Floer have to be made.
In Section 2, we deal with the transversality problem, i.e. we show that $\widehat{\mathcal{M}}(c^-, c^+, m)$ is a manifold for generic J. We generalize this to setups where everything is symmetric with respect to a symplectic symmetry of finite order.
In Section 3 we present many different properties of the moduli spaces. Due to the vastness of the topic, we do not prove the compactness of $\mathcal{M}(c^-, c^+, m)$. In Section 3.1, we merely provide some estimates which should ensure that the usual compactification due to Gromov works.
In Section 3.2, we show that the Rabinowitz-Floer homology does not depend on the auxiliary choices made for its definition. Even more surprising, we show that $RFH(W, \Sigma)$ actually is invariant under Liouville isomorphisms, thus giving an invariant of the filling by W of (Σ, ξ) (in the sense of Definition 9). In Proposition 64 we then see that under some circumstances $RFH(W, \Sigma)$ is even independent of W and hence yields an invariant of the contact structure.
Also in Section 3.2, we show that the action $\mathcal{A}^H$ induces a filtration of the complex $\big(RFC(H, h), \partial^F\big)$. We use this filtration to define truncated homology groups $RFH^{(a,b)}(W, \Sigma)$ and growth rates $\Gamma(W, \Sigma)$. The latter can be used to obtain more information on $RFH(W, \Sigma)$ if it is of infinite dimension. The Sections 3.3 and 3.4 are devoted to the Conley-Zehnder index and the $\mathbb{Z}$-grading of RFH.
In Section 4, we provide some useful facts about direct and inverse limits. In 4.3, Theorem 85, we show that $RFH(W, \Sigma)$ over field coefficients can be calculated using the singular homology $H_*(\mathcal{N}^\eta)$ without knowing an explicit Morse function h on $\mathcal{N}^\eta$ simply by algebraically pretending that we have a perfect Morse function h. This is purely algebraic and as a consequence less intuitive. For a first reading, it can be skipped, as we apply its main result in this thesis solely in situations where we could use standard arguments from spectral sequences.
In Sections 5 and 6 we introduce symplectic (co)homology and contact surgery (which includes the connected sum construction). In particular, we give a more detailed (and slightly corrected) proof of the fact that symplectic (co)homology is invariant under subcritical surgery, originally due to K. Cieliebak, [12]. We make this detour in order to show that Rabinowitz-Floer homology is also invariant under subcritical surgery, which we could not prove directly.[1]
In Section 7, we introduce the Brieskorn manifolds Σ_a and calculate $RFH_*(W_\varepsilon, \Sigma_a)$ explicitly for some Σ_a with fillings W_ε. In 7.3 we then prove our Main Theorem and some corollaries.
The appendices deal with technical details which are needed in the text but which are to long or to far of the general discourse. In Appendix A, we give an explicit example of

[1] In a recently published article, [17], Rem. 9.15, A. Oancea and K. Cieliebak show this invariance within the context of RFH.

a Morse-Smale pair on the unit cotangent bundle S^*S^n which is symmetric with respect to a certain involution. Appendix B recollects facts about convolutions. In Appendix C, we show that transversality always holds along constant solutions of (3). Finally in Appendix D, we give a short list of all the major assumptions and conventions that we use in this thesis.

The main result of this dissertation is the following theorem which states the existence of a rich variety of fillable contact structures or fillings on a differentiable manifold Σ, with $\dim \Sigma \geq 5$, if it supports at least one fillable contact structure.

Theorem (Main Theorem)**.**
Suppose that Σ is a differentiable manifold, $\dim \Sigma = 2n-1 \geq 5$, which supports at least one fillable contact structure with filling for which the conditions (A) and (B) are true. Then Σ satisfies at least one of the following alternatives:

a) For every fillable contact structure ξ on Σ and any filling W of (Σ, ξ), which satisfies (A) and (B), holds true that

$$\dim_{\mathbb{Z}_2} RFH_*(W, (\Sigma, \xi)) = \infty \qquad\qquad \forall * \in \mathbb{Z} \setminus [-n+1, n].$$

b) There is (at least) one contact structure on Σ for which there exist infinitely many different fillings.

c) There exist infinitely many different fillable contact structures on Σ.

Note the difference to dimension 3, where according to Eliashberg (see [26]) the only fillable contact structure on S^3 is the standard one and where due to Gromov, [28], the only filling for the standard contact structure on S^3 is the unit ball (B^4, ω_{std}). In particular, (S^3, ξ_{std}) does not satisfy a), b) or c).
In Section 7.3, we also prove the following dynamical and contact topological consequences.

Corollary.

- *If Σ satisfies alternative a) of the Main Theorem, then every fillable contact structure on Σ has for any generic contact form simple Reeb trajectories of arbitrary length.*

- *If Σ satisfies alternative b) but not a) of the Main Theorem, then there is at least one contact structure on Σ which has simple closed Reeb trajectories of arbitrary length for every generic contact form.*

Corollary. *Every Brieskorn manifold Σ_a supports at least 2 non-contactomorphic, exactly fillable contact structures.*

2. Transversality

The aim of this section is to show that $\widehat{\mathcal{M}}(c^-, c^+, m)$ is a finite dimensional manifold. The Subsections 2.1 through 2.3 provide analytic properties that will be needed in the proof of the fundamental Global Transversality Theorem 38 in 2.4.
Throughout this part, we assume (V, λ) to be an exact symplectic manifold that is the completion of a compact Liouville domain $\tilde{V}$ with contact boundary M. In particular, the symplectization $M \times [0, \infty)$ embeds into V and $V \setminus (M \times (0, \infty)) = \tilde{V}$ is compact. Moreover, we assume that $\Sigma \subset V$ is an exact contact hypersurface bounding a compact Liouville domain W and H a defining Hamiltonian for Σ.

2.1. The action functional

In this subsection, we will analyse more closely the Rabinowitz action functional $\mathcal{A}^H$ that we introduced in 1.5. We calculate its gradient $\nabla \mathcal{A}^H$ and Hessian $\nabla^2 \mathcal{A}^H$. We show that $\nabla^2 \mathcal{A}^H$ is a Fredholm operator of index zero and use this to show that (MB) implies that $\mathcal{A}^H$ is a Morse-Bott functional, i.e. that $crit\big(\mathcal{A}^H\big)$ is a submanifold of $\mathscr{L} \times \mathbb{R}$. In particular, we show that the critical manifolds $\mathcal{N}^\eta$ are isolated in the loop space $\mathscr{L} \times \mathbb{R}$. In classical Morse theory this would follow from the Morse Lemma. However, as $\mathscr{L} \times \mathbb{R}$ is infinite dimensional, there is no analogue of the Morse Lemma. Recall that $\mathcal{A}^H$ is given by

$$\mathcal{A}^H : \mathscr{L} \times \mathbb{R} \to \mathbb{R}, \qquad \mathcal{A}^H(v, \eta) := \int_0^1 \lambda(\dot{v}(t)) - \eta H(v(t)) dt,$$

where v is a smooth loop on V. More generally, we do not only consider smooth loops, but also loops that are of Sobolev-class $W^{k,p}$, $k \in \mathbb{N}$, $1 < p < \infty$, where a map $v : S^1 \to V$ is called $W^{k,p}$ if it is $W^{k,p}$ in every chart. We denote the space of such loops by $W^{k,p}(S^1, V)$ and remark that $W^{k,p}(S^1, V)$ is a Banach manifold while $H^k(S^1, V) = W^{k,2}(S^1, V)$ is even a Hilbert manifold (see [31], 2.3 ff.). The tangent space $T_v W^{k,p}(S^1, V)$ at any $v \in W^{k,p}(S^1, V)$ is given by the $W^{k,p}$-vector space of S^1-sections of v^*TV, i.e. a tangent vector $\mathfrak{v}$ at v is a 1-periodic $W^{k,p}$-map

$$\mathfrak{v} : S^1 \to TV \qquad \text{satisfying} \qquad \mathfrak{v}(t) \in T_{v(t)} V.$$

A metric on $W^{k,p}(S^1, V) \times \mathbb{R}$ is obtained via a family of ω-compatible almost complex structures $J_t(\cdot, n)$ depending on $(t, n) \in S^1 \times \mathbb{R}$. Given $(v, \eta) \in W^{k,p}(S^1, V) \times \mathbb{R}$ and $(\mathfrak{v}_1, \hat{\eta}_1), (\mathfrak{v}_2, \hat{\eta}_2) \in T_{(v,\eta)}(W^{k,p}(S^1, V) \times \mathbb{R})$ the metric g is defined by

$$g\left(\begin{pmatrix} \mathfrak{v}_1 \\ \hat{\eta}_1 \end{pmatrix}, \begin{pmatrix} \mathfrak{v}_2 \\ \hat{\eta}_2 \end{pmatrix} \right) := \int_0^1 \omega\Big(\mathfrak{v}_1(t)\, ,\, J_t\big(v(t), \eta\big)\mathfrak{v}_2(t)\Big) dt + \hat{\eta}_1 \cdot \hat{\eta}_2. \tag{6}$$

In the following, we are going to calculate the first and second variation of $\mathcal{A}^H$ in the form of gradient $\nabla \mathcal{A}^H$ and Hessian $\nabla^2 \mathcal{A}^H$ with respect to g. For that, assume that $(v_s, \eta_s) \subset W^{k,p} \times \mathbb{R}$, $k \geq 1$, is a differentiable 1-parameter family depending on $s \in (-\varepsilon, \varepsilon)$. Write

$$\frac{d}{dt} v = \dot{v}, \qquad \frac{d}{ds} v \bigg|_{s=0} = \mathfrak{v} \quad \text{and} \quad \frac{d}{ds} \eta \bigg|_{s=0} = \hat{\eta}.$$

Let ∇ denote the Levi-Civita connection of the metric $g_{t,n} = \omega(\cdot, J_{t,n}\cdot)$ for fixed (t, n). Observe that, when considered as a derivation of functions, we have

$$\frac{d}{dt} = \nabla_{\dot v} \qquad \text{and} \qquad \frac{d}{ds} = \nabla_{\mathfrak{v}}.$$

Note furthermore that $[\mathfrak{v}, \dot v] = 0$, as for any function $f \in C^\infty(V)$ holds

$$\nabla_{\mathfrak{v}}\nabla_{\dot v} f = \frac{d}{ds}\frac{d}{dt} f(v(s,t)) = \frac{d}{dt}\frac{d}{ds} f(v(s,t)) = \nabla_{\dot v}\nabla_{\mathfrak{v}} f.$$

Additionally, we have $\int_0^1 \frac{d}{dt}\lambda(\mathfrak{v}(t))dt = 0$, as $\mathfrak{v}$ is 1-periodic. This stated, we calculate the first variation of $\mathcal{A}^H$ in the direction of $(\mathfrak{v}, \hat\eta)$ as

$$\begin{aligned}
\nabla_{(\mathfrak{v},\hat\eta)}\mathcal{A}^H(v,\eta) &= \frac{d}{ds}\mathcal{A}^H(v_s,\eta_s)\bigg|_{s=0} \\
&= \int_0^1 \frac{d}{ds}\lambda(\dot v) - \hat\eta H(v) - \eta dH(\mathfrak{v})\,dt \\
&= \int_0^1 d\lambda(\mathfrak{v}, \dot v) + \frac{d}{dt}\lambda(\mathfrak{v}) + \lambda([\mathfrak{v}, \dot v]) - \hat\eta H(v) - d\lambda(\mathfrak{v}, \eta X_H)\,dt \\
&= \int_0^1 d\lambda(\mathfrak{v}, \dot v - \eta X_H) - \hat\eta H(v)\,dt \\
&= \int_0^1 \omega(\mathfrak{v}, J(-J)(\dot v - \eta X_H)\,dt - \hat\eta \cdot \int_0^1 H(v)\,dt \\
&= g\left(\begin{pmatrix}\mathfrak{v} \\ \hat\eta\end{pmatrix}, \begin{pmatrix}-J(\dot v - \eta X_H) \\ -\int_0^1 H(v)\,dt\end{pmatrix}\right).
\end{aligned}$$

The gradient of $\mathcal{A}^H$ with respect to g is hence given by

$$\nabla\mathcal{A}^H = -\begin{pmatrix} J(\dot v - \eta X_H) \\ \int_0^1 H(v)dt \end{pmatrix}. \tag{7}$$

This shows, that Definition 14 really defines $\nabla\mathcal{A}^H$-gradient trajectories. As stated before, the critical points (v, η) of $\mathcal{A}^H$ are solutions of the equations

$$\begin{aligned}
& 0 = \dot v - \eta X_H \qquad \text{and} \qquad 0 = \textstyle\int_0^1 H(v(t))dt \\
\Leftrightarrow\quad & \dot v = \eta R \qquad \text{and} \qquad im(v) \subset \Sigma
\end{aligned}$$

Recall that this implied that $crit\big(\mathcal{A}^H\big) = \mathcal{P}(\alpha)$ and that it followed from Lemma 16 that $\mathcal{A}^H\big(crit\big(\mathcal{A}^H\big)\big) = spec(\Sigma, \alpha)$. Note in particular, that all critical points are in $\mathscr{L} \times \mathbb{R}$, i.e. all critical v are smooth even if we consider $\mathcal{A}^H$ on $W^{k,p}(S^1, V)$.

Next, we are going to calculate the Hessian of $\mathcal{A}^H$ at a critical point $(v, \eta) \in \mathscr{L} \times \mathbb{R}$, where $\nabla\mathcal{A}^H(v, \eta) = 0$. Actually, we calculate the self-adjoint operator

$$\nabla^2\mathcal{A}^H_{(v,\eta)} : T_{(v,\eta)}\Big(W^{k,p}(S^1, V) \times \mathbb{R}\Big) \to T_{(v,\eta)}\Big(W^{k-1,p}(S^1, V) \times \mathbb{R}\Big),$$

which satisfies for $x, y \in T_{(v,\eta)}\big(W^{k,p}(S^1, V) \times \mathbb{R}\big)$ that

$$Hess\,\mathcal{A}^H(x, y) = \nabla_x\, g(y, \nabla\mathcal{A}^H) = g(y, \nabla^2\mathcal{A}^H x).$$

Note that $g(\nabla_x y, \nabla\mathcal{A}^H) = 0$ at a critical point and hence $\nabla^2\mathcal{A}^H_{(v,\eta)}x = \nabla_x\nabla\mathcal{A}^H(v,\eta)$. This implies that $Hess\,\mathcal{A}^H$ and $\nabla^2\mathcal{A}^H$ are symmetric, i.e. $g(y,\nabla^2\mathcal{A}^H x) = g(\nabla^2\mathcal{A}^H y, x)$, as shown by the following short calculation

$$\begin{aligned} Hess\,\mathcal{A}^H_{(v,\eta)}(x,y) = & \quad \nabla_x\, g(y,\nabla\mathcal{A}^H_{(v,\eta)}) = \nabla_x\nabla_y\mathcal{A}^H_{(v,\eta)} = \\ = & \quad \nabla_y\nabla_x\mathcal{A}^H_{(v,\eta)} + \nabla_{[x,y]}\mathcal{A}^H_{(v,\eta)} = \nabla_y\nabla_x\mathcal{A}^H_{(v,\eta)} = Hess\,\mathcal{A}^H_{(v,\eta)}(y,x). \end{aligned}$$

For the calculation of $\nabla^2\mathcal{A}^H$, we consider again a differentiable 1-parameter family $(v_s,\eta_s) \subset W^{k,p}\times\mathbb{R}$ with tangent vectors $(\mathfrak{v},\hat\eta) := (\mathfrak{v}_s,\hat\eta_s) \in T_{(v_s,\eta_s)}\big(W^{k,p}(S^1,V)\times\mathbb{R}\big)$. Then, we trivialize $(v_s,\eta_s)^*T\big(W^{k,p}(S^1,V)\times\mathbb{R}\big)$ over $\mathbb{R}\times S^1$, which allows us to calculate

$$\tfrac{d}{ds}\nabla\mathcal{A}^H(v_s,\eta_s)\big|_{s=0} = \nabla_{\mathfrak{v},\hat\eta}\nabla\mathcal{A}^H(v_s,\eta_s).$$

In the trivialization we differentiate $\nabla\mathcal{A}^H$ at any point (v,η), even non-critical ones! We use this result later in Section 2.4, where the derivative of $\nabla\mathcal{A}^H$ along (v_s,η_s) appears in the linearization of the Rabinowitz-Floer equation (3). Using (7), we calculate

$$\begin{aligned} \nabla_{(\mathfrak{v},\hat\eta)}\nabla\mathcal{A}^H(v,\eta) &= \nabla_{(\mathfrak{v},\hat\eta)}\begin{pmatrix} -J(\dot v-\eta X_H) \\ -\int_0^1 H(v)\,dt\end{pmatrix} \\ &= \begin{pmatrix} -\nabla_{\mathfrak{v}}\big(J(\dot v-\eta X_H)\big) - \hat\eta(\partial_n J)(\dot v-\eta X_H) + J\hat\eta X_H \\ -\int_0^1 dH(\mathfrak{v})\,dt\end{pmatrix} \\ \nabla_{\mathfrak{v}}\big(J(\dot v-\eta X_H)\big) &= \big(\nabla_{\mathfrak{v}}J\big)(\dot v-\eta X_H) - J(\nabla_{\mathfrak{v}}\dot v - \nabla_{\mathfrak{v}}\eta X_H) \\ &= \big(\nabla_{\mathfrak{v}}J\big)(\dot v-\eta X_H) + J\big(\nabla_{\dot v}\mathfrak{v} + [\mathfrak{v},\dot v] - \nabla_{\eta X_H}\mathfrak{v} - [\mathfrak{v},\eta X_H]\big) \\ &= \big(\nabla_{\mathfrak{v}}J\big)(\dot v-\eta X_H) + J\nabla_{(\dot v-\eta X_H)}\mathfrak{v} - J[\mathfrak{v},\eta X_H], \end{aligned}$$

where we used again $[\mathfrak{v},\dot v]=0$. The derivative of $\nabla\mathcal{A}^H$ along (v_s,η_s) is hence given by

$$\begin{pmatrix} -\Big(\nabla_{\mathfrak{v}}J + \hat\eta(\partial_n J)\Big)(\dot v-\eta X_H) - J\Big(\nabla_{(\dot v-\eta X_H)}\mathfrak{v} - [\mathfrak{v},\eta X_H] - \hat\eta X_H\Big) \\ -\int_0^1 dH(\mathfrak{v})dt\end{pmatrix}. \tag{8}$$

At a critical point, where $\dot v - \eta X_H = 0$, it takes the form

$$\nabla^2\mathcal{A}^H_{(v,\eta)}(\mathfrak{v},\hat\eta) = \begin{pmatrix} J\big([\mathfrak{v},\eta X_H] + \hat\eta X_H\big) \\ -\int_0^1 dH(\mathfrak{v})dt\end{pmatrix}. \tag{9}$$

Note that this expression does not depend on the family (v_s,η_s), but only on $(\mathfrak{v},\hat\eta)$ – its derivative at $s=0$. Let ϕ denote the Reeb flow on Σ. As X_H coincides with the Reeb field on Σ, we have that the flow ϕ_H of ηX_H is given by $\phi_H^t = \phi^{\eta t}$. At a critical point, we can hence express the term $-[\mathfrak{v},\eta X_H] = [\eta X_H,\mathfrak{v}]$ with the help of ϕ as

$$[\eta X_H,\mathfrak{v}](t_0) = \mathcal{L}_{\eta X_H}\mathfrak{v}(t_0) = \frac{d}{dt}\left(\phi^{\eta t}\right)^*\mathfrak{v}(t_0) = \frac{d}{dt}\left(D\phi^{\eta t}\right)^{-1}\mathfrak{v}(t_0+t)\Big|_{t=0}.$$

Note that for $\eta=0$, this becomes the ordinary derivative $\frac{d}{dt}$ on $T_{v(0)}V$. Having this in mind, we could write symbolically $\nabla^2\mathcal{A}^H$ at a critical point as

$$\nabla^2\mathcal{A}^H_{(v,\eta)}(\mathfrak{v},\hat\eta) = \begin{pmatrix} J\big(-\frac{d}{dt}\mathfrak{v} + \hat\eta X_H\big) \\ -\int_0^1 dH(\mathfrak{v})dt\end{pmatrix}.$$

Lemma 20. *Assume that* (MB) *holds true. Then* $\ker \nabla^2 \mathcal{A}^H$ *consists of pairs* $(\mathfrak{v}, 0)$, *where* $\mathfrak{v}(t) \in T_{v(t)}\mathcal{N}^\eta$ *for all* t *and* $\mathfrak{v}$ *is constant with respect to the Reeb flow* ϕ, *i.e.*

$$\mathfrak{v}(t) = \big(D\phi^{\eta t}\big)\mathfrak{v}(0) \qquad \forall\, t.$$

In particular, $\ker \nabla^2 \mathcal{A}^H$ *is finite dimensional with* $\dim\big(\ker \nabla^2 \mathcal{A}^H_{(v,\eta)}\big) = \dim_{v(0)} \mathcal{N}^\eta$.

Proof: It follows from (9), that $(\mathfrak{v}, \hat{\eta}) \in \ker \nabla^2 \mathcal{A}^H_{(v,\eta)}$ if and only if

$$\text{I.} \quad \frac{d}{dt}\big(D\phi^{\eta t}\big)^{-1}\mathfrak{v}(t_0+t)\Big|_{t=0} = \hat{\eta} X_H(v(t_0)) \qquad \text{and} \qquad \text{II.} \quad 0 = \int_0^1 dH(\mathfrak{v})dt.$$

The proof has now 3 steps:

- First, we show that $\mathfrak{v}(t) \in T_{v(t)}\Sigma$ for all $t \in S^1$. Note that H, dH and X_H are invariant under the flow $\phi^{\eta t}$ of ηX_H. This implies that

$$dH_{v(t)}\big(\mathfrak{v}(t)\big) = dH_{v(0)}\left(\big(D\phi^{-\eta t}\big)\mathfrak{v}(t)\right)$$

and hence with I. that

$$\begin{aligned}\frac{d}{dt} dH_{v(t)}\big(\mathfrak{v}(t)\big)\Big|_{t=t_0} &= dH_{v(0)}\left(\frac{d}{dt}\big(D\phi^{-\eta(t+t_0)}\big)\mathfrak{v}(t+t_0)\Big|_{t=0}\right) \\ &= dH_{v(0)}\Big(D\phi^{-\eta t_0}\big(\hat{\eta} X_H(v(t_0))\big)\Big) = dH_{v(0)}\left(\hat{\eta} X_H(v(0))\right) = 0.\end{aligned}$$

Thus, we have that $dH(\mathfrak{v}(t))$ is constant. Therefore, equation II. implies that

$$0 = \textstyle\int_0^1 dH(\mathfrak{v}(t))dt = \int_0^1 dH(\mathfrak{v}(t_0))dt = dH(\mathfrak{v}(t_0)) \qquad \forall\, t_0 \in S^1.$$

As Σ is a regular level set of H, we have $\ker dH_{v(t)} = T_{v(t)}\Sigma$ and hence that $\mathfrak{v}(t) \in T_{v(t)}\Sigma$ for all t.

- Next, we show $\hat{\eta} = 0$. Assume first that $\eta \neq 0$. We then obtain from equation II.

$$\begin{aligned}0 &= \int_0^1 dH(\mathfrak{v})\,dt = \int_0^1 d\lambda(\mathfrak{v}, X_H)\,dt \\ &= \int_0^1 \partial_{\mathfrak{v}}\lambda(X_H) - \partial_{X_H}\lambda(\mathfrak{v}) - \lambda\big([\mathfrak{v}, X_H]\big)\,dt = \int_0^1 0 - \frac{1}{\eta}\cdot\frac{d}{dt}\lambda(\mathfrak{v}) + \frac{\hat{\eta}}{\eta}\,dt = \frac{\hat{\eta}}{\eta}.\end{aligned}$$

Here, we used equation I. and $\frac{d}{dt} = \partial_{\dot{v}} = \eta\partial_{X_H}$ and that $\lambda(\mathfrak{v})$ is 1-periodic and $\partial_{\mathfrak{v}}\lambda(X_H) = 0$, as $\lambda(X_H) = 1$ is constant throughout Σ and $\mathfrak{v} \in T\Sigma$. Hence $\hat{\eta} = 0$ if $\eta \neq 0$.
Now, if $\eta = 0$, then v is constant and equation I. becomes the linear differential equation $\partial_t\mathfrak{v}(t_0) = \hat{\eta} X_H(v(0))$ for a map $\mathfrak{v} : S^1 \to T_{v(0)}V$. The solutions to this problem are of the form $\mathfrak{v}(t) = \mathfrak{v}(0) + t\cdot\hat{\eta} X_H(v(0))$. However, we require $\mathfrak{v}(1) = \mathfrak{v}(0)$ and this holds if and only if $\hat{\eta} = 0$, as $X(v(0)) \neq 0$.

- Finally, we prove the lemma. As $\hat{\eta} = 0$, equation I. yields

$$\begin{aligned} & \frac{d}{dt}\left(D\phi^{\eta t}\right)^{-1}\mathfrak{v}(t_0+t)\Big|_{t=0} = 0 && \forall t_0 \in \mathbb{R} \\ \Leftrightarrow \quad & \left(D\phi^{-\eta t}\right)\mathfrak{v}(t) = \mathfrak{v}(0) && \forall t \in \mathbb{R} \\ \Leftrightarrow \quad & \mathfrak{v}(t) = (D\phi^{\eta t})\mathfrak{v}(0) && \forall t \in \mathbb{R} \end{aligned}$$

Recall that $\mathfrak{v}$ has to be 1-periodic. However, not every solution of the last equation satisfies this. It holds if and only if $\mathfrak{v}(0) = \mathfrak{v}(1) = D\phi^{\eta}\mathfrak{v}(0)$, in other words if and only if $\mathfrak{v}(0) \in \ker(D\phi^{\eta} - id) = T_{v(0)}\mathcal{N}^{\eta}$, by assumption (MB). $\square$

In the following, let $(v, \eta) \in crit\big(\mathcal{A}^H\big)$ and $1 < p < \infty$ be fixed and abbreviate

$$W^{k,p} := W^{k,p}(v, \eta) := T_{(v,\eta)}\left(W^{k,p}(S^1, V) \times \mathbb{R}\right)$$

for the $W^{k,p}$-vector fields along (v, η). Similarly write $L^p = L^p(v, \eta)$ and $C^{\infty} := C^{\infty}(v, \eta)$ for the L^p- resp. C^{∞}-vector fields. Let us also abbreviate $K := K(v, \eta) := \ker \nabla^2\mathcal{A}^H_{(v,\eta)}$. Note that in this terminology the operator $\nabla^2\mathcal{A}^H_{(v,\eta)}$ maps $W^{k+1,p}$ to $W^{k,p}$ as in the term $[X_H, \mathfrak{v}]$ the vector field $\mathfrak{v}$ is differentiated once. Now, for any $k \geq 0$ and $1 < p < \infty$, the following two lemmas will show that the cokernel of $\nabla^2\mathcal{A}^H$ equals K, thus showing that $\nabla^2\mathcal{A}^H$ is self-adjoint and a Fredholm operator of index 0.

Lemma 21. *Assume that (MB) holds. Then it holds that the image $\nabla^2\mathcal{A}^H\big(W^{k+1,p}\big)$ is closed in $W^{k,p}$ for all $p \geq 1$.*

Proof:
We will show that $\nabla^2\mathcal{A}^H$ has on its image a continuous right inverse, i.e. there exists a continuous operator $U : \nabla^2\mathcal{A}^H(W^{k+1,p}) \to W^{k+1,p}$ such that $\nabla^2\mathcal{A}^H \circ U = Id$. Given such a U, we prove the lemma as follows: If $(\mathfrak{w}_n, \xi_n) \subset \nabla^2\mathcal{A}^H(W^{k+1,p})$ is a sequence which converges in $W^{k,p}$ to some $(\mathfrak{w}, \xi)$, then by continuity of U and completeness of $W^{k+1,p}$ there exists $(\mathfrak{v}, \hat{\eta})$ such that $U(\mathfrak{w}_n, \xi_n) \to (\mathfrak{v}, \hat{\eta})$. Then

$$(\mathfrak{w}, \xi) = \lim_{n\to\infty}(\mathfrak{w}_n, \xi_n) = \lim_{n\to\infty}\nabla^2\mathcal{A}^H(U(\mathfrak{w}_n, \xi_n)) = \nabla^2 A^H(\mathfrak{v}, \eta),$$

by continuity of $\nabla^2\mathcal{A}^H$, which shows that $\nabla^2\mathcal{A}^H(W^{k+1,p})$ is closed in $W^{k,p}$.

To construct U, note that the flow $\phi^{\eta t}$ of ηX_H provides a trivialization Φ of v^*TV

$$\Phi : v^*TV \to [0,1] \times T_{v(0)} \qquad \big(v(t), \xi(t)\big) \mapsto \Big(t, \left(D\phi^{\eta t}\right)^{-1}\xi(t)\Big).$$

We can then express $\nabla^2\mathcal{A}^H$ in this coordinates as

$$\nabla^2\mathcal{A}^H(\mathfrak{v}, \hat{\eta}) = -\begin{pmatrix} I \cdot \left(\frac{d}{dt}\mathfrak{v} - \hat{\eta}X\right) \\ \int_0^1 dH_{v(0)}\big(\mathfrak{v}\big)dt \end{pmatrix}, \tag{10}$$

where $X(t)=X_H(v(0))$ is a constant vector and $I(t)=\big(D\phi^{\eta t}\big)^{-1}J_{v(t)}\big(D\phi^{\eta t}\big)$ a bounded path of matrices with $I^2=-Id$. Note that $\mathfrak{v}$ is here a map from $[0,1]$ to $T_{v(t_0)}$. In order to guarantee that $\Phi^{-1}\mathfrak{v}$ is 1-periodic, we have to require that $D\phi^{\eta}\big(\mathfrak{v}(1)\big)=\mathfrak{v}(0)$.
Recall that $D\phi^{\eta}$ restricted to $T_{v(0)}\mathcal{N}^{\eta}$ is the identity and let C denote any complement to $T_{v(0)}\mathcal{N}^{\eta}$ in $T_{v(0)}V$. Then $\big((D\phi^{\eta})^{-1}-\mathbb{1}\big)$ restricted to C is invertible. Let E denote the linear map

$$E: T_{v(0)}V=T_{v(0)}\mathcal{N}^{\eta}\oplus C\to T_{v(0)}\mathcal{N}^{\eta}\oplus C,\qquad E=0\oplus\pi_C\Big(\big(D\phi^{\eta}\big)^{-1}-\mathbb{1}\Big)^{-1}.$$

For $(\mathfrak{w},\xi)=\nabla^2\mathcal{A}^H(\mathfrak{v},\hat\eta)\in\nabla^2\mathcal{A}^H\big(W^{k+1,p}\big)$, we construct $U(\mathfrak{w},\xi)$ using the ansatz

$$\mathfrak{w}=-I\cdot\left(\frac{d}{dt}\mathfrak{v}-\hat\eta X\right)\qquad\text{and}\qquad \xi=-\int_0^1 dH_{v(0)}\big(\mathfrak{v}\big)dt.$$

Using the symplectic form $\omega=\omega_{v(0)}$ on $T_{v(0)}V$, we solve the first equation for $\hat\eta$ to get

$$\widetilde\eta:=-\omega\Big(IX,\textstyle\int_0^1 I\mathfrak{w}\,dt\Big).$$

Note that $\widetilde\eta=\hat\eta$. Hence, we can use the fundamental theorem of calculus to solve the first equation for $\mathfrak{v}$ to

$$\widetilde{\mathfrak{v}}(t):=\textstyle\int_0^t(I\mathfrak{w}+\widetilde\eta X)dt+E\big(\int_0^1(I\mathfrak{w}+\widetilde\eta X)dt\big).$$

The E-term is needed, as we shall see, to ensure that $D\phi^{\eta}\big(\widetilde{\mathfrak{v}}(1)\big)=\widetilde{\mathfrak{v}}(0)$. Assuming this, we have that

$$U:\nabla^2\mathcal{A}^H\big(W^{k+1,p}\big)\to W^{k+1,p},\qquad (\mathfrak{w},\xi)\mapsto(\widetilde{\mathfrak{v}},\widetilde\eta)$$

is a well-defined linear operator. The continuity of U is obvious. To see that U is a right inverse of $\nabla^2\mathcal{A}^H$, i.e. $\nabla^2\mathcal{A}^H\circ U=Id$ on $im\big(\nabla^2\mathcal{A}^H\big)$, let $(\mathfrak{w},\xi)=\nabla^2\mathcal{A}^H(\mathfrak{v},\hat\eta)$ and calculate

$$\begin{aligned}\widetilde\eta=-\omega\Big(IX,\int_0^1 I\mathfrak{w}\,dt\Big)&=-\omega\Big(IX,\int_0^1 I\circ -I\big(\tfrac{d}{dt}\mathfrak{v}-\hat\eta X\big)\,dt\Big)\\&=-\omega\Big(IX,\int_0^1\tfrac{d}{dt}\mathfrak{v}-\hat\eta X dt\Big)\\&=-\omega\Big(IX,\mathfrak{v}(1)-\mathfrak{v}(0)-\hat\eta X\Big)=\hat\eta,\end{aligned}$$

where the last line follows as $D\phi^{\eta}X=X$ so that the X-part of $\mathfrak{v}(1)-\mathfrak{v}(0)$ is zero. Moreover

$$\begin{aligned}\widetilde{\mathfrak{v}}(t)&=\int_0^t I\circ -I\big(\tfrac{d}{dt}\mathfrak{v}-\hat\eta X\big)+\hat\eta X\,dt+E\int_0^1 I\circ -I\big(\tfrac{d}{dt}\mathfrak{v}-\hat\eta X\big)+\hat\eta X\,dt\\&=\int_0^t\tfrac{d}{dt}\mathfrak{v}\,dt+E\int_0^1\tfrac{d}{dt}\mathfrak{v}\,dt\\&=\mathfrak{v}(t)-\mathfrak{v}(0)+E\big(\mathfrak{v}(1)-\mathfrak{v}(0)\big)\\&=\mathfrak{v}(t)-\mathfrak{v}(0)+E\big((D\phi^1)^{-1}\mathfrak{v}(0)-\mathfrak{v}(0)\big)\\&=\mathfrak{v}(t)-\mathfrak{v}(0)+E\big((D\phi^1)^{-1}-\mathbb{1})\mathfrak{v}(0)\big)\\&=\mathfrak{v}(t)-\pi_{T_{v(0)}\mathcal{N}^{\eta}}\big(\mathfrak{v}(0)\big).\end{aligned}$$

Here, the last line implies that $(\widetilde{\mathfrak{v}},\widetilde{\eta})$ is in fact $(\mathfrak{v},\hat{\eta})$ minus $\big(\pi_{T_{v(0)}\mathcal{N}^\eta}\big(\mathfrak{v}(0)\big),0\big)$, which is an element in the kernel of $\nabla^2\mathcal{A}^H$. This shows that $\nabla^2\mathcal{A}^H(\widetilde{\mathfrak{v}},\widetilde{\eta}) = \nabla^2\mathcal{A}^H(\mathfrak{v},\hat{\eta}) = (\mathfrak{w},\xi)$, i.e. $\nabla^2\mathcal{A}^H\circ U = Id$. It also shows that $D\phi^\eta\big(\widetilde{\mathfrak{v}}(1)\big) = \widetilde{\mathfrak{v}}(0)$, as $D\phi^\eta|_{T\mathcal{N}^\eta} = Id$ and hence

$$D\phi^\eta\big(\widetilde{\mathfrak{v}}(1)\big) = D\phi^\eta\Big(\mathfrak{v}(1) - \pi_{T_{v(0)}\mathcal{N}^\eta}\big(\mathfrak{v}(0)\big)\Big) = \mathfrak{v}(0) - \pi_{T_{v(0)}\mathcal{N}^\eta}\big(\mathfrak{v}(0)\big) = \widetilde{\mathfrak{v}}(0). \qquad \square$$

Lemma 22. *Assume that* (MB) *holds. Then*

$$W^{k,p} = \ker\nabla^2\mathcal{A}^H \oplus \nabla^2\mathcal{A}^H\big(W^{k+1,p}\big).$$

In particular $\operatorname{coker}\nabla^2\mathcal{A}^H = \ker\nabla^2\mathcal{A}^H$ *and* $\nabla^2\mathcal{A}^H$ *is a Fredholm operator of index 0.*

Proof: Let $\perp$ denote the L^2-orthogonal complement with respect to the metric g. As we integrate over the compact 1-dimensional domain S^1, it follows from Rellich's Theorem that $W^{k,p}$ embeds into L^2 for $k\geq 1$ and all $p\in(1,\infty)$ or for $k=0$ and $p\geq 2$. Thus it makes sense for these k and p to consider the closed space

$$\big(\nabla^2\mathcal{A}^H(W^{k+1,p})\big)^\perp \subset L^2.$$

<u>*Claim*</u> *:* $\big(\nabla^2\mathcal{A}^H(W^{k+1,p})\big)^\perp = \ker\nabla^2\mathcal{A}^H$.

We prove this claim below, but first let us show how the claim implies the lemma.
For $k\geq 1$ or $k=1$ and $p\geq 2$, we argue as follows. As $\ker\nabla^2\mathcal{A}^H$ consists of smooth elements, we find that

$$\ker\nabla^2\mathcal{A}^H = \ker\nabla^2\mathcal{A}^H\cap W^{k,p} = \big(\nabla^2\mathcal{A}^H(W^{k+1,p})\big)^\perp\cap W^{k,p}.$$

As $\nabla^2\mathcal{A}^H(W^{k+1,p})$ is closed in $W^{k,p}$, we hence have that

$$W^{k,p} = \ker\nabla^2\mathcal{A}^H \oplus \nabla^2\mathcal{A}^H(W^{k+1,p}).$$

For $k=0$ and $1<p\leq 2$, we use the dual space $(L^p)^* = L^q$, where $1/p+1/q=1$ so that $q\geq 2$ and L^q embedds into L^2. Then we consider the annihilator $\big(\nabla^2\mathcal{A}^H(W^{1,p})\big)^0\subset L^q$ and find again (by repeating the proof of the claim) that

$$\big(\nabla^2\mathcal{A}^H(W^{1,p})\big)^0 = \ker\nabla^2\mathcal{A}^H$$

and hence $L^p = \ker\nabla^2\mathcal{A}^H\oplus\nabla^2\mathcal{A}^H(W^{1,p})$.

Proof of the claim:
Basically, the statement follows from elliptic regularity as $\nabla^2\mathcal{A}^H$ is an elliptic operator of order 1. However, we give here for convenience an explicit proof. Recall from the proof of Lemma 21 that the flow $\phi^{\eta t}$ of ηX_H provides a trivialization of $v^*TV\cong[0,1]\times T_{v(0)}V$ under which elements of $W^{k,p}$ become $W^{k,p}$-maps $\mathfrak{v}:[0,1]\to T_{v(0)}V$ such that $D\phi^\eta\big(\mathfrak{v}(1)\big) = \mathfrak{v}(0)$. We expressed $\nabla^2\mathcal{A}^H$ in this trivialization by (10), where I is a t-dependent matrix such that $I^2=-Id$. As ω is invariant under the flow $\phi^{\eta t}$, we find that the metric g on $[0,1]\times T_{v(0)}$ is just

$$g\big((\mathfrak{v}_0,\hat{\eta}_0),(\mathfrak{v}_1,\hat{\eta}_1)\big) = \int_0^1\omega_{v(0)}\big(\mathfrak{v}_0,I\mathfrak{v}_1\big)dt + \hat{\eta}_0\cdot\hat{\eta}_1.$$

In this framework, we can now describe $\big(\nabla^2\mathcal{A}^H(W^{k+1,p})\big)^\perp$. It is obvious that $\ker\nabla^2\mathcal{A}^H \subset \big(\nabla^2\mathcal{A}^H(W^{k+1,p})\big)^\perp$. For the opposite inclusion let $(\mathfrak{v},\hat\eta)$ be any element in the complement. We then have for all $(\mathfrak{w},\xi)\in W^{k+1,p}$ with $D\phi^\eta\big(\mathfrak{w}(1)\big)=\mathfrak{w}(0)$ that

$$0 = g\left(\begin{pmatrix}\mathfrak{v}\\ \hat\eta\end{pmatrix}, \nabla^2\mathcal{A}^H\begin{pmatrix}\mathfrak{w}\\ \xi\end{pmatrix}\right) \overset{(10)}{=} \int_0^1 \omega\big(\mathfrak{v}, \tfrac{d}{dt}\mathfrak{w} - \xi\cdot X\big)dt + \hat\eta\cdot\int_0^1 dH(\mathfrak{w})dt.$$

Considering in particular $\xi=0$ and $\xi=1$, we find that this holds if and only if

$$0 = \int_0^1 \omega\big(\mathfrak{v}, \tfrac{d}{dt}\mathfrak{w}\big)dt + \hat\eta\cdot\int_0^1 \omega\big(\mathfrak{w}, X\big)dt \qquad \text{and} \qquad 0 = \int_0^1 \omega\big(\mathfrak{v}, X\big)dt.$$

As the Liouville vector field Y_λ is also preserved under the flow of ηX_H (at least on Σ), we find that $\mathfrak{w}_0(t) := Y_\lambda(v_0)$ satisfies $D\phi^\eta\big(\mathfrak{w}_0(1)\big)=\mathfrak{w}_0(0)$. For this map we have $\frac{d}{dt}\mathfrak{w}_0 = 0$ and hence we conclude that

$$0 = \hat\eta\cdot\int_0^1 \omega(\mathfrak{w}_0, X)dt = \hat\eta\cdot\int_0^1 \omega(Y_\lambda, X)dt = \hat\eta.$$

Thus, we find that $\mathfrak{v}$ has to satisfy the following two equations:

$$I.\quad 0 = \int_0^1 \omega(\mathfrak{v}, X)dt \qquad \text{and} \qquad II.\quad 0 = \int_0^1 \omega\big(\mathfrak{v}, \tfrac{d}{dt}\mathfrak{w}\big)dt,$$

where $\mathfrak{w}$ may still be any $W^{k+1,p}$-map $\mathfrak{w}:[0,1]\to T_{v(0)}V$ with $D\phi^{\eta t}\big(\mathfrak{w}(1)\big)=\mathfrak{w}(0)$. The second condition on $\mathfrak{w}$ is automatically satisfied if $\operatorname{supp}\mathfrak{w}\subset(0,1)$. In particular, we can take $\mathfrak{w}=\rho_\delta*\mathfrak{u}$, where $\mathfrak{u}$ is any test-function with support in $(\delta,1-\delta)$ and ρ_δ is a smooth bump function with $\operatorname{supp}\rho_\delta\subset(-\delta,\delta)$, $\int_{-\infty}^\infty \rho_\delta\,dt = 1$ and $\rho_\delta(t)=\rho_\delta(-t)$ (see Appendix B). Then we have from II. and Corollary 125 that

$$\begin{aligned}0 = \int_0^1 \omega\big(\mathfrak{v}, \tfrac{d}{dt}(\rho_\delta*\mathfrak{u})\big)dt = \int_0^1 \omega\big(\mathfrak{v}, (\tfrac{d}{dt}\rho_\delta)*\mathfrak{u}\big)dt &= \int_0^1 \omega\big((\tfrac{d}{dt}\rho_\delta)*\mathfrak{v}, \mathfrak{u}\big)dt\\ &= \int_0^1 \omega\big(\tfrac{d}{dt}(\rho_\delta*\mathfrak{v}), \mathfrak{u}\big)dt.\end{aligned}$$

As this equation holds for any test function $\mathfrak{u}$ with $\operatorname{supp}\mathfrak{u}\subset(\delta,1-\delta)$, we infer that $\rho_\delta*\mathfrak{v}$ is constant on $(\delta,1-\delta)$. As $\rho_\delta*\mathfrak{v}\to\mathfrak{v}$ in L^p (see Lemma 122), we conclude that $\mathfrak{v}$ is also constant and hence smooth. From equation I., we then conclude that the Y_λ-component of $\mathfrak{v}$ has to be zero. Now as $D\phi^\eta\big(\mathfrak{v}(0)\big) = D\phi^\eta\big(\mathfrak{v}(1)\big)=\mathfrak{v}(0)$, we know that $\mathfrak{v}(t)=\mathfrak{v}(0)\in\ker(D\phi^\eta - Id)$ and this implies that $(\mathfrak{v},\xi)=(\mathfrak{v},0)$ lies in $\ker\nabla^2\mathcal{A}^H$. □

Theorem 23. *If (MB) is satisfied, then the following equivalent statements are true:*

- *The spectrum* $spec(\Sigma,\alpha)=\mathcal{A}^H\big(\mathcal{P}(\alpha)\big)$ *is closed and discrete.*
- $\mathcal{P}(\alpha)=crit\big(\mathcal{A}^H\big)$ *is a submanifold of* $\mathscr{L}\times\mathbb{R}$ *with disjoint components* $\mathcal{N}^\eta$*, i.e.* $\mathcal{A}^H$ *is a Morse-Bott functional.*
- *For any real numbers* $-\infty<a<b<\infty$ *there are only finitely many* $\eta\in[a,b]$ *such that* $\mathcal{N}^\eta\neq\emptyset$.

Proof: The equivalence of the 3 statements is obvious. We therefore only show the first one. That $spec(\Sigma, \alpha)$ is closed can be seen as follows: Suppose that $(\eta_n) \subset spec(\Sigma, \lambda)$ is a sequence with $\lim \eta_n = \eta$. This means that there exists a sequence of Reeb trajectories (v_n) whose periods are η_n. As Σ is compact, we may assume by the Arzela-Ascoli theorem that v_n converges uniformly to a Reeb trajectory v, whose period has to be η. Hence $\eta \in \mathrm{spec}(\Sigma, \lambda)$.

The proof of $\mathrm{spec}(\Sigma, \lambda)$ being discrete is more involved. In what follows, it suffices to restrict to the Hilbert spaces $H^k = W^{k,2}$. It follows from Lemma 20 and 22 that $\nabla^2 \mathcal{A}^H : H^1 \to L^2$ is a Fredholm operator of index zero. Let again $K = \ker \nabla^2 \mathcal{A}^H = \operatorname{coker} \nabla^2 \mathcal{A}^H$. We abbreviate for a moment the orthogonal complements of K in H^1 resp. L^2 by $S := K^{\perp_{H^1}} \subset H^1$ and $R := K^{\perp_{L^2}} \subset L^2$. The restriction $\nabla^2 \mathcal{A}^H : S \to R$ is bijective and due to the Open Mapping Theorem therefore a (continuous) isomorphism. This implies in particular that $\nabla^2 \mathcal{A}^H$ is bounded from below, i.e. there exists a constant $C > 0$, depending continuously on (v, η), such that

$$||\nabla^2 \mathcal{A}^H(\mathfrak{v}, \hat{\eta})||^2_{L^2} \geq C \cdot ||(\mathfrak{v}, \hat{\eta})||^2_{H^1},$$

for all $(\mathfrak{v}, \hat{\eta}) \in S$. If we denote by $\pi : H^1 \to S$ the orthogonal projection onto S, we may rewrite this more elegantly as

$$||\nabla^2 \mathcal{A}^H(\mathfrak{v}, \hat{\eta})||^2_{L^2} \geq C \cdot ||\pi(\mathfrak{v}, \hat{\eta})||^2_{H^1}$$

for all $(\mathfrak{v}, \hat{\eta}) \in H^1$. As C depends continuously on (v, η), it can be chosen globally for all $(v, \eta) \in \mathcal{N}^\eta$, as $\mathcal{N}^\eta \subset \Sigma$ is closed and hence compact. Now consider the function

$$e : H^k(S^1, V) \times \mathbb{R} \to \mathbb{R}, \qquad e(x) = ||\nabla \mathcal{A}^H(x)||^2_{L^2}$$

on the Hilbert manifold $H^k(S^1, V) \times \mathbb{R}$ of pairs (v, η) consisting of H^k-loops v and real numbers η. We find that $e(x) = 0$ if and only if $x \in crit\big(\mathcal{A}^H\big)$. Moreover, for $x \in crit\big(\mathcal{A}^H\big)$ and $X, Y \in T_x\big(H^k(S^1, V) \times \mathbb{R}\big)$, one easily calculates that

$$e(x) = 0, \qquad De(x) = 0 \qquad \text{and} \qquad D^2 e(x)[X, Y] = 2g\big(\nabla^2 \mathcal{A}^H(x) X, \nabla^2 \mathcal{A}^H(x) Y\big).$$

Let $(v_a, \eta_a) \subset H^k(S^1, V) \times \mathbb{R}$ be a k-times differentiable family in a such that $(v_0, \eta_0) \in \mathcal{N}^{\eta_0} \subset crit\big(\mathcal{A}^H\big)$ and $\frac{d}{da}(v_a, \eta_a)|_{a=0} = (\mathfrak{v}, \hat{\eta})$. Then, the Taylor formula at (v_0, η_0) yields

$$e(v_a, \eta_a) = ||\nabla^2 \mathcal{A}^H_{(v_0, \eta_0)}(\mathfrak{v}, \hat{\eta})||^2_{L^2} \cdot a^2 + O(a^3).$$

Using the above estimate, we find constants $c, d > 0$ such that at least for small a we have

$$e(v_a, \eta_a) \geq c \cdot a^2 \left(||\pi(\mathfrak{v}, \hat{\eta})||^2_{L^2} - d \cdot a\right). \tag{11}$$

Note that c and d depend again continuously on (v, η) and may therefore be chosen globally on $\mathcal{N}^\eta$. If $(\mathfrak{v}, \hat{\eta}) \notin \ker \nabla^2 \mathcal{A}^H$, it follows that $e(v_a, \eta_a) > 0$ near (not at) $a = 0$ and hence that (v_0, η_0) is the only critical point of $\mathcal{A}^H$ on (v_a, η_a) near $a = 0$.

We recall that $\mathcal{N}^\eta \subset H^k(S^1, V) \times \mathbb{R}$ is assumed to be a submanifold. Note that $H^k(v, \eta)$

and $K(v,\eta)$ are the tangent spaces to $H^k(S^1,V)\times\mathbb{R}$ resp. $\mathcal{N}^\eta$ at points $(v,\eta)\in\mathcal{N}^\eta\subset H^k(S^1,V)\times\mathbb{R}$. According to Theorem 1.3.5 in [31], there exists in Hilbert manifolds around each point (v,η) in a submanifold $\mathcal{N}^\eta$ a submanifold chart, i.e. there exists a closed linear subspace $E\subset H^k(v,\eta)$ such that $H^k(v,\eta)=K(v,\eta)\oplus E$ and open neighborhoods $U\subset H^k(S^1,V)$ around (v,η), $V'\subset K(v,\eta), V''\subset E$ each around 0 together with a diffeomorphism

$$\phi: V'\times V''\to U \qquad \text{with} \qquad \phi(V'\times\{0\})=\mathcal{N}^\eta\cap U.$$

Fix $y\in V'$. Then any $(\mathfrak{v},\hat{\eta})\in E$ with $||(\mathfrak{v},\hat{\eta})||^2_{L^2}=1$ yields a well-defined path

$$(v_a,\eta_a):=\phi\big(y+a\cdot(\mathfrak{v},\hat{\eta})\big)$$

at least for $|a|<\delta_E$, with δ_E so small that $B_{\delta_E}(0)\subset V''$. Due to the following Lemma 24, there exists a constant $k>0$, such that $||\pi(\mathfrak{v},\hat{\eta})||^2_{L^2}>k$ for all $(\mathfrak{v},\hat{\eta})\in E$. Set $\varepsilon_E=\min\{\delta_E,k/d\}$, with d being the constant from (11). Then we find that the only critical point on (v_a,η_a) for $|a|<\varepsilon_E$ is $\phi(y)\in\mathcal{N}^\eta$ at $a=0$. Hence we see for the open set $\phi\big(V'\times B_{\varepsilon_E}(0)\big)\subset H^k(S^1,V)\times\mathbb{R}$ that

$$\phi\big(V'\times B_{\varepsilon_E}(0)\big)\cap crit\big(\mathcal{A}^H\big)=\mathcal{N}^\eta\cap U.$$

By covering $\mathcal{N}^\eta$ with a finite number of charts ϕ, we obtain that $\mathcal{N}^\eta\subset crit\big(\mathcal{A}^H\big)$ is isolated. $\square$

Lemma 24. *Let H be a Hilbert space, $K\subset H$ a finite dimensional subspace and $E\subset H$ a closed subspace such that $H=K\oplus E$, i.e. E is any closed complement of K. Denote by $\pi: H\to K^\perp$ the orthogonal projection onto the orthogonal complement of K. Then there exists a constant $k>0$, depending on E, such that for all $x\in E$ holds*

$$||\pi(x)||^2\geq k\cdot||x||^2.$$

Proof: Obviously, it suffice to show the claim for all $x\in E$ with $||x||^2=1$. Assume the contrary. Then there exists a sequence $(x_n)\subset E$, $||x_n||^2=1$ with $\lim||\pi(x_n)||^2=0$. Note that

$$||x_n-\pi(x_n)||^2=||x_n||^2-||\pi(x_n)||^2=1-||\pi(x_n)||^2\leq 1,$$

as π is an orthogonal projection. Hence, we see that $x_n-\pi(x_n)\in B_1(0)\subset K$ lies in the unit ball of K, which is compact, as K has finite dimension. By considering a subsequence, still denoted by x_n, we may assume that $x_n-\pi(x_n)$ converges in K to $y\in K$. Now we have

$$||x_n-y||\leq||x_n-\pi(x_n)-y||+||\pi(x_n)||\to 0.$$

In other words, x_n converges in H to $y\in K$. As E is closed, this means that $y\in E$. Hence $y\in E\cap K=\{0\}$ and therefore $y=0$, contradicting $||y||=\lim||x_n||=1$. Thus, the assumption is false and the lemma follows. $\square$

2.2. Asymptotic estimates

In this section, we show the following theorem, whose most important statement is that convergent $\mathcal{A}^H$-gradient flow lines always converge exponentially.

Theorem 25. *Let $(v, \eta) : \mathbb{R} \times S^1 \to V \times \mathbb{R}$ be a solution of the Rabinowitz-Floer equation (3). Then the following statements are equivalent:*

1. *$E(v, \eta) < \infty$ and (v, η) stays in a compact region in $V \times \mathbb{R}$.*
2. *$|\partial_s(v, \eta)(s, t)| \to 0$ and dist.$\big((v, \eta)(s, t), \mathcal{N}^{\eta^\pm}\big) \to 0$ for some $\eta^\pm \in$ spec(Σ, α), where both limits are uniform in t for $s \to \pm\infty$.*
3. *There exist constants $\delta, c > 0$ and $v^\pm \in \mathcal{N}^{\eta^\pm}$ such that $|\partial_s(v, \eta)(s, t)| \leq c \cdot e^{-\delta|s|}$ and dist.$\Big(\left(\begin{smallmatrix} v(t) \\ \eta \end{smallmatrix}\right)(s), \left(\begin{smallmatrix} v^\pm(t) \\ \eta^\pm \end{smallmatrix}\right)\Big) \leq c \cdot e^{-\delta|s|}$ for all $(s, t) \in \mathbb{R} \times S^1$.*

Remark.

- The term $E(v, \eta)$ is the energy of (v, η) as defined in Definition 17 by

$$E(v, \eta) = \int_{\mathbb{R} \times S^1} \left|\partial_s \left(\begin{smallmatrix} v \\ \eta \end{smallmatrix}\right)(s, t)\right|^2 dt\, ds = \int_{-\infty}^{\infty} ||\nabla \mathcal{A}^H(v, \eta)||^2 ds = \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}^H(v, \eta)\, ds.$$

  Here, we denote for $(Y, n) \in L^2(S^1, TV) \times L^2(\mathbb{R})$ by $\left|\left(\begin{smallmatrix} Y \\ n \end{smallmatrix}\right)\right|^2$ the pointwise norm given by $\omega(Y, JY) + n^2$, while $\left|\left|\left(\begin{smallmatrix} Y \\ n \end{smallmatrix}\right)\right|\right|^2$ is either the L^2-norm $\int_0^1 |Y|^2 dt + n^2$ over S^1 or by abuse of notation the L^2-norm $\int_{-\infty}^{+\infty} \big(\int_0^1 |Y|^2 dt + n^2\big)\, ds$ over $\mathbb{R} \times S^1$.

- It follows from 3. using local coordinates that we have for any convergent $\mathcal{A}^H$-gradient trajectory (v, η) and any $p > 1$

$$\begin{aligned} \big(v(s, t), \eta(s)\big) - \big(v^\pm(t), \eta^\pm\big) &\in W^{1,p}(\mathbb{R}; e^{r|s|} dt\, ds) \\ \partial_s(v, \eta) &\in L^p(\mathbb{R}; e^{r|s|} dt\, ds) \end{aligned}$$

  for r small enough. For the precise definition of these weighted Sobolev spaces, see Definition 31 below.

The proof of Theorem 25 is given in several steps. It is obvious that 3. implies 1. and the verification is left to the reader. The proof that 1. implies 2. is given in Proposition 26. It uses many ideas from Salamon, [46]. That 2. implies 3. is proved in Proposition 32 and is inspired by a similar proof due to Bourgeois and Oancea in [7].

Proposition 26. *Let (v, η) be an $\mathcal{A}^H$-gradient trajectory with $E(v, \eta) < \infty$ and (v, η) staying in a compact region $W \subset V \times \mathbb{R}$. Then, $\lim_{s \to \pm\infty} \partial_s(v, \eta) = 0$ and there exists $\eta^\pm \in$ spec(Σ, α) with dist.$\Big((v, \eta)(s), \mathcal{N}^{\eta^\pm}\Big) \to 0$ and all limits are uniform in t.*

Proof: We only prove the case $s \to +\infty$, as $s \to -\infty$ is completely analogue.

1. First, we show that $E(v, \eta) < \infty$ and (v, η) staying in a compact set implies that $\lim_{s\to\infty} \partial_s(v, \eta) = 0$ uniformly in t. The proof relies on the following a priori estimate:

$$\begin{aligned} &\int_{B_r(s,t)} \left|\partial_s \begin{pmatrix} v \\ \eta \end{pmatrix}\right|^2 < \delta \quad \forall t \in S^1 \\ \Rightarrow \quad &\exists\, t^* \in S^1 : \forall t \in S^1 : \ \left|\partial_s \begin{pmatrix} v \\ \eta \end{pmatrix}(s,t)\right|^2 \le \frac{Ar^2}{2} + \frac{8}{\pi r^2} \int_{B_r(s,t^*)} \left|\partial_s \begin{pmatrix} v \\ \eta \end{pmatrix}\right|^2 \end{aligned} \tag{12}$$

   for solutions of the Rabinowitz-Floer equation (3) which stay in a compact region $W \subset V \times \mathbb{R}$. Here, $A > 0$ and $\delta > 0$ are constants depending on W, ω, J and H, but not on (v, η). This estimate is proven in Lemma 27 and 28. Assuming (12), we show the uniform convergence as follows: As $E(v, \eta)$ is finite and the energy density $|\partial_s \left(\begin{smallmatrix} v \\ \eta \end{smallmatrix}\right)|^2$ always non-negative, we can choose for any ε with $0 < \varepsilon < \sqrt{\delta}$ an $s_0 > 0$ so large such that

$$\int_{s_0}^{\infty} \int_0^1 \left|\partial_s \begin{pmatrix} v \\ \eta \end{pmatrix}\right|^2 \, dt\, ds \le \varepsilon^2.$$

   Then we may apply (12) with $r = \sqrt{\varepsilon}$ to obtain $|\partial_s \left(\begin{smallmatrix} v \\ \eta \end{smallmatrix}\right)(s,t)|^2 \le (A/2 + 8/\pi)\,\varepsilon$ for $s \ge s_0 + \sqrt{\varepsilon}$. This shows that $\partial_s \left(\begin{smallmatrix} v \\ \eta \end{smallmatrix}\right)$ converges to zero uniformly as $s \to \infty$.

2. It remains to show that $(v, \eta)(s)$ lies uniformly arbitrarily close to $crit\big(\mathcal{A}^H\big)$ as $s \to \infty$. Recall that

$$\partial_s \begin{pmatrix} v \\ \eta \end{pmatrix}(s,t) = -\begin{pmatrix} J(\partial_t v - \eta X_H \\ \int_0^1 H(v)dt \end{pmatrix}.$$

   As $\partial_s \left(\begin{smallmatrix} v \\ \eta \end{smallmatrix}\right) \to 0$ uniformly, we find in particular that $\partial_t v - \eta X_H(v)$ converges uniformly to zero. As (v, η) stays in the compact region W, we find that v and η stay in compact regions and that $\partial_t v$ is bounded. We may hence apply the Arzela-Ascoli Theorem which shows that we can extract from any sequence $s_k \to \infty$ a subsequence (still denoted s_k), such that $(v, \eta)(s_k)$ converges uniformly to some $(v^+, \eta^+) \in crit\big(\mathcal{A}^H\big)$. All possible values for η^+ have to lie in $spec(\Sigma, \alpha) = \mathcal{A}^H\big(crit\big(\mathcal{A}^H\big)\big)$. As $spec(\Sigma, \alpha)$ is closed and discrete, we conclude that all $\eta(s_k)$ converge to the same η^+, which shows that $\eta(s) \to \eta^+$. A similar argument shows that for every sequence (s_k) such that $v(s_k)$ converges holds that the limit is a point on $\mathcal{N}^{\eta^+}$. Using some auxiliary metric on V, we find that this implies that $dist.\big((v, \eta)(s,t), \mathcal{N}^{\eta^+}\big) \to 0$ uniformly in t. □

To complete the proof of Proposition 26, it remains to show (12). We will do so by applying the following general a priori estimate (13) to $w = |\partial_s \left(\begin{smallmatrix} v \\ \eta \end{smallmatrix}\right)|^2$. The estimate (13) is a variation on a similar estimate by Dietmar Salamon (see [46], Prop. 1.21). The main difference is that in Rabinowitz-Floer theory the Laplacian Δw is not bounded from below by a pointwise constraint $-A - Bw^2$, but by $-A - B(w^2 + \int_{S^1} w^2)$, which involves the values of w on a whole circle. This is due to the fact that the Rabinowitz-Floer equation is only semi-local. The t^*-shifted center of integration on the right side of our estimate (13) pays tribute to this.

Lemma 27. *Assume that $w : \mathbb{R}^2 \to \mathbb{R}$ is a C^2-function satisfying*

- $w \geq 0$.
- $w(s,t) = w(s, t+T)$ *for some* $T > 0$ *and all* $(s,t) \in \mathbb{R}^2$.
- $\Delta w \geq -A - B\big(w^2 + \frac{1}{T}\int_0^T w^2 dt\big)$,
  where $\Delta w = \partial_s^2 w + \partial_t^2 w$ *is the Laplacian and* $A, B \geq 0$ *are constants.*
- $\displaystyle\int_{B_r(0,t)} w \leq \frac{\pi}{32B} \qquad \forall\, t \in [0,T]$,
  where $B_r(0,t) \subset \mathbb{R}^2$ *is the closed ball with radius* r *around the point* $(0,t)$.

Then there exists $t^* \in [0,T]$ *such that for all* t *holds*

$$w(0,t) \leq \frac{Ar^2}{2} + \frac{8}{\pi r^2}\int_{B_r(0,t^*)} w. \tag{13}$$

Proof: Following [36], the proof is devided into five steps.

Step 1: *The lemma holds (even stronger) with $B = 0$, i.e. if $\Delta w \geq -A$ then*

$$w(s_0,t_0) \leq \frac{Ar^2}{8} + \frac{1}{\pi r^2}\int_{B_r(s_0,t_0)} w \qquad \forall\,(s_0,t_0) \in \mathbb{R}^2.$$

This is the mean value inequality for the subharmonic function

$$\widetilde{w}(s,t) = w(s,t) + A\frac{(s-s_0)^2 + (t-t_0)^2}{4}.$$

For completeness, we give the following proof from [36]. Assume without loss of generality that $s_0 = t_0 = 0$. Then by the divergence theorem, we have

$$0 \leq \frac{1}{\rho}\int_{B_\rho(0)} \Delta\widetilde{w} = \frac{1}{\rho}\int_{\partial B_\rho(0)} \frac{\partial \widetilde{w}}{\partial \nu} = \int_0^{2\pi} \frac{d}{d\rho}\widetilde{w}(\rho e^{i\theta})d\theta = \frac{d}{d\rho}\left(\frac{1}{\rho}\int_{\partial B_\rho(0)} \widetilde{w}\right),$$

where ν is the outer normal vector field. Hence, we have for $0 < \rho < r$

$$\frac{1}{2\pi\rho}\int_{\partial B_\rho(0)} \widetilde{w} \leq \frac{1}{2\pi r}\int_{\partial B_r(0)} \widetilde{w}.$$

The term on the left converges to $\widetilde{w}(0)$ as ρ tends to zero. Hence

$$2\pi r \widetilde{w}(0) \leq \int_{\partial B_r(0)} \widetilde{w}.$$

Integrating this inequality from 0 to r gives the mean value inequality for $\widetilde{w}$. Hence

$$w(0) = \widetilde{w}(0) \leq \frac{1}{\pi r^2} \int_{B_r(0)} \widetilde{w} = \frac{Ar^2}{8} + \frac{1}{\pi r^2} \int_{B_r(0)} w.$$

<u>Step 2</u>: *It suffices to prove the lemma for $r = 1$*

Suppose that w satisfies the assumptions and define $\widetilde{w}$ and $\widetilde{A}, \widetilde{B}$ and $\widetilde{T}$ by

$$\widetilde{w}(s,t) := w(rs, rt), \quad \widetilde{A} := A \cdot r^2, \quad \widetilde{B} := B \cdot r^2 \quad \text{and} \quad \widetilde{T} := \frac{1}{r} T.$$

Then $\widetilde{w}$ is positive, $\widetilde{T}$-periodic and we have

$$\begin{aligned} \Delta \widetilde{w} &= r^2 \Delta w \geq -r^2 A - r^2 B \cdot \left(w^2 + \tfrac{1}{T} \textstyle\int_0^T w^2 \, dt \right) \\ &= -\widetilde{A} - \widetilde{B} \left(\widetilde{w}^2 + \tfrac{1}{\widetilde{T}} \textstyle\int_0^{\widetilde{T}} \widetilde{w}^2 \, dt \right) \end{aligned}$$

and

$$\int_{B_1(0,t)} \widetilde{w} = \frac{1}{r^2} \int_{B_r(0,rt)} w \leq \frac{\pi}{32 \widetilde{B}} \quad \forall \, t.$$

Hence, assuming the lemma for $r = 1$, we obtain

$$w(0,t) = \widetilde{w}(0, \tfrac{1}{r} t) \leq \frac{\widetilde{A}}{2} + \frac{8}{\pi} \int_{B_1(0,t^*)} \widetilde{w} = \frac{Ar^2}{2} + \frac{8}{\pi r^2} \int_{B_r(0,rt^*)} w.$$

<u>Step 3</u>: *It suffices to prove the lemma for $B = 1$.*

Suppose that w satisfies the assumptions and define $\widetilde{w}$ and $\widetilde{A}$ by

$$\widetilde{w}(s,t) := B \cdot w(s,t), \quad \widetilde{A} = B \cdot A.$$

Then $\widetilde{w}$ is still positive and T periodic. Moreover

$$\Delta \widetilde{w} \geq -\widetilde{A} - \left(\widetilde{w}^2 + \tfrac{1}{T} \textstyle\int_0^T \widetilde{w}^2 \, dt \right) \qquad \text{and} \qquad \textstyle\int_{B_1(0,t)} \widetilde{w} \leq \frac{\pi}{32} \quad \forall \, t.$$

Hence assuming the lemma for $B = 1$, we obtain

$$w(0,t) = \tfrac{1}{B} \widetilde{w}(0,t) \leq \frac{\frac{1}{B} \widetilde{A}}{2} + \frac{8}{\pi} \int_{B_1(0,t^*)} \tfrac{1}{B} \widetilde{w} = \frac{A}{2} + \frac{8}{\pi} \int_{B_1(0,t^*)} w.$$

<u>Step 4</u>: *The Heinz Trick: Assume $B = r = 1$ and define $f : [0,1] \times [0,T] \to \mathbb{R}$ by*

$$f(\rho, t) = (1 - \rho)^2 \cdot \max_{B_\rho(0,t)} w.$$

As f is non-negative, continuous and $f(1,t) = 0$, there exists $\rho^* \in [0,1)$ and $t^* \in [0,T]$ and $c > 0$ and $z^* \in B_{\rho^*}(0,t^*)$ such that

$$f(\rho^*, t^*) = \max_{\rho, \, t} f(\rho, t) = (1 - \rho^*)^2 \cdot c, \qquad c = w(z^*) = \max_{B_{\rho^*}(0,t^*)} w.$$

Write $\varepsilon = \frac{1-\rho^*}{2}$. *For* $0 \le \rho \le \varepsilon$ *then holds that*

$$w(z^*) = c \le \frac{A\rho^2}{8} + 4c^2\rho^2 + \frac{1}{\rho^2\pi}\int_{B_1(0,t^*)} w. \qquad (*)$$

To see this, note that $\rho^* + \varepsilon = \frac{1+\rho^*}{2} < 1$. Since $B_\varepsilon(z) \subset B_{\rho^*+\varepsilon}(0,t)$ for all $z = (s,t) \in [-\rho^*, \rho^*] \times [0,T]$, it holds for these z that

$$\max_{B_\varepsilon(z)} w \le \max_{B_{\rho^*+\varepsilon}(0,t)} w = \frac{f(\rho^*+\varepsilon,t)}{(1-\rho^*-\varepsilon)^2} = \frac{4f(\rho^*+\varepsilon,t)}{(1-\rho^*)^2} \le \frac{4f(\rho^*,t^*)}{(1-\rho^*)^2} = 4c. \qquad (**)$$

$$\Rightarrow \qquad \Delta w \ge -A - w^2 - \tfrac{1}{T}\int_0^T w^2 dt \ge -A - 16c^2 - \tfrac{T}{T}16c^2 = -A - 32c^2.$$

Now, $(*)$ follows from step 1 at $w(z^*) = c$ with $r = \rho \le \varepsilon$ and A replaced by $A + 32c^2$. (Note that $B_\rho(z^*) \subset B_1(0,t^*)$!)

Step 5: *The lemma holds for* $r = 1$ *and* $B = 1$.

If $c \le A/8$, then $w(0,t) \le 4c \le A/2$ by $(**)$ and this implies the lemma. Hence we may assume that $c \ge A/8$. We prove that this implies $4c \cdot \varepsilon^2 \le \frac{1}{2}$. Suppose otherwise that $\varepsilon^2 \ge \frac{1}{8c}$. Then in $(*)$, we can choose $\rho = \sqrt{\frac{1}{8c}} \le \varepsilon$ and obtain

$$\begin{aligned}
c &\le \frac{A}{8}\cdot\frac{1}{8c} + 4c^2\cdot\frac{1}{8c} + \frac{8c}{\pi}\int_{B_1(0,t^*)} w \\
&\le c\cdot\varepsilon^2 + \frac{c}{2} + \frac{8c}{\pi}\int_{B_1(0,t^*)} w \\
\Leftrightarrow \qquad \frac{c\cdot\left(\frac{1}{2}-\varepsilon^2\right)\pi}{8c} &\le \int_{B_1(0,t^*)} w \\
\Rightarrow \qquad \frac{\pi}{32} &\le \int_{B_1(0,t^*)} w,
\end{aligned}$$

where we used that $\varepsilon = \frac{1-\rho^*}{2} \le \frac{1}{2}$. But the last inequality is a contradiction to the fourth assumption. Hence $4c\cdot\varepsilon^2 \le \frac{1}{2}$. Now consider $(*)$ with $\rho = \varepsilon$

$$\begin{aligned}
c &\le \frac{A\varepsilon^2}{8} + 4c^2\varepsilon^2 + \frac{1}{\varepsilon^2\pi}\int_{B_1(0,t^*)} w \qquad \le \frac{A}{32} + \frac{c}{2} + \frac{1}{\varepsilon^2\pi}\int_{B_1(0,t^*)} w \\
\Rightarrow \qquad \frac{c}{2} &\le \frac{A}{32}\cdot\frac{1}{4\varepsilon^2} + \frac{1}{\varepsilon^2\pi}\int_{B_1(0,t^*)} w \\
\Rightarrow \qquad 4c\varepsilon^2 &\le \frac{A}{16} + \frac{8}{\pi}\int_{B_1(0,t^*)} w \qquad \le \frac{A}{2} + \frac{8}{\pi}\int_{B_1(0,t^*)} w.
\end{aligned}$$

Hence $$w(0,t) = f(0,t) \le f(\rho^*,t^*) = (1-\rho^*)^2\cdot c = 4\varepsilon^2\cdot c \le \frac{A}{2} + \frac{8}{\pi}\int_{B_1(0,t^*)} w. \qquad \square$$

Lemma 28. *Assume that $(v,\eta) \in C^\infty(\mathbb{R}\times S^1, V)\times C^\infty(\mathbb{R},\mathbb{R})$ is a solution of the Rabinowitz-Floer equation (3) which stays in a compact region $W \subset V\times\mathbb{R}$. Then there exist constants $A,\delta>0$ depending on W and ω, H, J such that*

$$\int_{B_r(s,t)} \left|\partial_s \begin{pmatrix} v\\ \eta\end{pmatrix}\right|^2 < \delta \quad \forall t\in S^1$$

$$\textit{implies} \quad \exists\, t^*\in S^1 \;:\; \left|\partial_s \begin{pmatrix} v\\ \eta\end{pmatrix}(s,t)\right|^2 \leq \frac{Ar^2}{2} + \frac{8}{\pi r^2}\int_{B_r(s,t^*)} \left|\partial_s \begin{pmatrix} v\\ \eta\end{pmatrix}\right|^2 \quad \forall t\in S^1.$$

Remark. The Maximum Principle, Proposition 90, and the a priori estimates on η, Corollary 51, show that W only depends on $E(v,\eta)$ (and on H_s in the homotopy case).

Proof: As announced, we prove the statement by applying Lemma 27 to the function

$$w:\mathbb{R}\times S^1\to\mathbb{R} \qquad w(s,t) := \frac{1}{2}\left|\partial_s \begin{pmatrix} v\\ \eta\end{pmatrix}(s,t)\right|^2.$$

As $w\geq 0$ and w being 1-periodic is obvious, we only have to show that there exist constants $A,B>0$ such that

$$\Delta w \geq -A - B\big(w^2 + \textstyle\int_0^1 w^2 dt\big).$$

The constant δ is then given by $\pi/16B$. This estimate is somewhat technical but straightforward. The trick is that we can use the equation $\partial_s v = -J(\partial_t - \eta X_H)$ to rewrite the second order terms in Δw as first order derivatives of J and X_H. Let us abbreviate $\mathfrak{v} := \partial_s v$, $\dot v := \partial_t v$, $\dot\eta = \partial_s\eta$ and $X := X_H$. Then, the Rabinowitz-Floer equation (3) translates to

$$\mathfrak{v} = -J(\dot v - \eta X) \quad\Leftrightarrow\quad \dot v = J\mathfrak{v} + \eta X, \qquad \dot\eta = \textstyle\int_0^1 H(v)dt.$$

As $\frac{d}{ds}\frac{d}{dt}f(v) = \frac{d}{dt}\frac{d}{ds}f(v)$ for every function f, we have as in Section 2.1 that, $[\mathfrak{v},\dot v]=0$ and hence $\nabla_{\mathfrak{v}}\dot v = \nabla_{\dot v}\mathfrak{v}$, where ∇ denotes the Levi-Civita connection of the metric $g=\omega(\cdot, J\cdot)$. Now, w is given by $w(s,t) = \frac{1}{2}\big(|\mathfrak{v}|^2+\dot\eta^2\big) = \frac{1}{2}\big(g(\mathfrak{v},\mathfrak{v})+\dot\eta^2\big)$ and the Laplacian satisfies

$$\Delta w = |\nabla_{\mathfrak{v}}\mathfrak{v}|^2 + (\partial_s\dot\eta)^2 + |\nabla_{\dot v}\mathfrak{v}|^2 + \kappa.$$

Here, κ is an error term given by

$$\kappa = g(\mathfrak{v}, \nabla_{\mathfrak{v}}\nabla_{\mathfrak{v}}\mathfrak{v}) + g(\mathfrak{v},\nabla_{\dot v}\nabla_{\dot v}\mathfrak{v}) + \dot\eta\cdot(\partial_s\partial_s\dot\eta).$$

In the following, we estimate the components of κ. First we have

$$\begin{aligned}
\nabla_{\mathfrak{v}}\mathfrak{v} &= -\nabla_{\mathfrak{v}}\big(J(\dot v-\eta X)\big) = -(\nabla_{\mathfrak{v}}J)(\dot v - \eta X) - J(\nabla_{\mathfrak{v}}\dot v - \dot\eta X - \eta\nabla_{\mathfrak{v}}X)\\
\nabla_{\dot v}\dot v &= \nabla_{\dot v}\big(J\mathfrak{v}+\eta X\big) = (\nabla_{\dot v}J)\mathfrak{v} + J(\nabla_{\dot v}\mathfrak{v}) + \eta\nabla_{\dot v}X\\
\nabla_{\mathfrak{v}}\dot v &= \nabla_{\mathfrak{v}}\big(J\mathfrak{v}+\eta X\big) = (\nabla_{\mathfrak{v}}J)\mathfrak{v} + J(\nabla_{\mathfrak{v}}\mathfrak{v}) + \dot\eta X + \eta\nabla_{\mathfrak{v}}X
\end{aligned}$$

which implies with $\nabla_{\mathfrak{v}}\dot{v} = \nabla_{\dot{v}}\mathfrak{v}$ that

$$\nabla_{\mathfrak{v}}\mathfrak{v} + \nabla_{\dot{v}}\dot{v} = (\nabla_{\dot{v}}J)\mathfrak{v} - (\nabla_{\mathfrak{v}}J)(\dot{v} - \eta X) + J(\dot{\eta}X + \eta\nabla_{\mathfrak{v}}X) + \eta\nabla_{\dot{v}}X \tag{$*$}$$

which implies

$$\begin{aligned}\nabla_{\mathfrak{v}}\big(\nabla_{\mathfrak{v}}\mathfrak{v} + \nabla_{\dot{v}}\dot{v}\big) = \quad & (\nabla_{\dot{v}}J)\nabla_{\mathfrak{v}}\mathfrak{v} - (\nabla_{\mathfrak{v}}J)(\nabla_{\mathfrak{v}}\dot{v} - \dot{\eta}X - \eta\nabla_{\mathfrak{v}}X)\\ & + (\nabla_{\mathfrak{v}}\nabla_{\dot{v}}J)\mathfrak{v} - (\nabla_{\mathfrak{v}}\nabla_{\mathfrak{v}}J)(\dot{v} - \eta X)\\ & + (\nabla_{\mathfrak{v}}J)(\dot{\eta}X + \eta\nabla_{\mathfrak{v}}X) + J\big((\partial_s\dot{\eta})X + \dot{\eta}\nabla_{\mathfrak{v}}X + \dot{\eta}\nabla_{\mathfrak{v}}X + \eta\nabla_{\mathfrak{v}}\nabla_{\mathfrak{v}}X\big)\\ & + \dot{\eta}\nabla_{\dot{v}}X + \eta\nabla_{\mathfrak{v}}\nabla_{\dot{v}}X.\end{aligned}$$

Additionally, we have

$$\nabla_{\mathfrak{v}}\nabla_{\mathfrak{v}}\mathfrak{v} + \nabla_{\dot{v}}\nabla_{\dot{v}}\mathfrak{v} = \nabla_{\mathfrak{v}}\big(\nabla_{\mathfrak{v}}\mathfrak{v} + \nabla_{\dot{v}}\dot{v}\big) + \nabla_{\dot{v}}\nabla_{\mathfrak{v}}\dot{v} - \nabla_{\mathfrak{v}}\nabla_{\dot{v}}\dot{v} = \nabla_{\mathfrak{v}}\big(\nabla_{\mathfrak{v}}\mathfrak{v} + \nabla_{\dot{v}}\dot{v}\big) + R(\dot{v}, \mathfrak{v}; \dot{v}),$$

where R is the curvature tensor of ∇. Note that $im(v, \eta)$ lying in the compact region W guarantees that the number η and the tensor fields $J, \nabla J, \nabla\nabla J, X, \nabla X$ and R are bounded. Moreover, we have for any tensor T the estimate

$$|\nabla_{\mathfrak{v}}\nabla_{\mathfrak{v}}T| \leq ||\nabla\nabla T|| \cdot |\mathfrak{v}|^2 + ||\nabla T|| \cdot |\nabla_{\mathfrak{v}}\mathfrak{v}|.$$

Using this, the expression for $\nabla_{\mathfrak{v}}\dot{v}$ calculated above and the estimate $|\dot{v}| \leq ||J|| \cdot |\mathfrak{v}| + |\eta X|$, we find that there exists a constant $\tilde{C} > 0$ (depending on H, ω, J on W) such that

$$|\nabla_{\mathfrak{v}}\nabla_{\mathfrak{v}}\mathfrak{v} + \nabla_{\dot{v}}\nabla_{\dot{v}}\mathfrak{v}| \leq \tilde{C}\Big(|\mathfrak{v}|\big(|\nabla_{\mathfrak{v}}\mathfrak{v}| + |\nabla_{\dot{v}}\mathfrak{v}| + |\dot{\eta}|\big) + |\nabla_{\mathfrak{v}}\mathfrak{v}| + |\mathfrak{v}|^3 + |\mathfrak{v}|^2 + |\mathfrak{v}| + |(\partial_s\dot{\eta})|\Big).$$

Using Young's inequality, we thus find constants $C, D > 0$ such that

$$\begin{aligned}& g\big(\mathfrak{v}, \nabla_{\mathfrak{v}}\nabla_{\mathfrak{v}}\mathfrak{v} + \nabla_{\dot{v}}\nabla_{\dot{v}}\mathfrak{v}\big)\\ & \geq -\tilde{C}\Big(\big(|\mathfrak{v}|^2 + |\mathfrak{v}|\big)|\nabla_{\mathfrak{v}}\mathfrak{v}| + |\mathfrak{v}|^2 \cdot |\nabla_{\dot{v}}\mathfrak{v}| + |\mathfrak{v}|^4 + |\mathfrak{v}|^3 + |\mathfrak{v}|^2 \cdot |\dot{\eta}| + |\mathfrak{v}| \cdot |(\partial_s\dot{\eta})|\Big)\\ & \geq -D - C\left|\begin{pmatrix}\mathfrak{v}\\ \dot{\eta}\end{pmatrix}\right|^4 - \frac{1}{2}\Big|\nabla_{\mathfrak{v}}\mathfrak{v}\Big|^2 - \frac{1}{2}\Big|\nabla_{\dot{v}}\mathfrak{v}\Big|^2.\end{aligned} \tag{$**$}$$

Next, we estimate the $\dot{\eta}$ part of κ. We have

$$\partial_s\partial_s\,\dot{\eta} = \partial_s\partial_s\int_0^1 H(v)\,dt = \partial_s\int_0^1 dH(\mathfrak{v})\,dt = \int_0^1 (\nabla_{\mathfrak{v}}dH)(\mathfrak{v}) + dH(\nabla_{\mathfrak{v}}\mathfrak{v})\,dt.$$

As $dH(\dot{v})$ is a 1-periodic function, we find

$$0 = dH(\dot{v})(1) - dH(\dot{v})(0) = \int_0^1 \frac{d}{dt}dH(\dot{v})\,dt = \int_0^1 (\nabla_{\dot{v}}dH)(\dot{v}) + dH(\nabla_{\dot{v}}\dot{v})\,dt$$

which implies with $(*)$ that

$$\begin{aligned}\partial_s\partial_s\,\dot{\eta} &= \int_0^1 (\nabla_{\mathfrak{v}}dH)(\mathfrak{v}) + dH(\nabla_{\mathfrak{v}}\mathfrak{v}) + (\nabla_{\dot{v}}dH)(\dot{v}) + dH(\nabla_{\dot{v}}\dot{v})\,dt\\ &= \int_0^1 \left(\begin{array}{c}(\nabla_{\mathfrak{v}}dH)(\mathfrak{v}) + (\nabla_{\dot{v}}dH)(\dot{v})\\ + dH\Big((\nabla_{\dot{v}}J)\mathfrak{v} - (\nabla_{\mathfrak{v}}J)(\dot{v} - \eta X) + J(\dot{\eta}X + \eta\nabla_{\mathfrak{v}}X) + \eta\nabla_{\dot{v}}X\Big)\end{array}\right) dt\end{aligned}$$

and hence there is another constant $\tilde{C}' > 0$ such that

$$|\dot{\eta} \cdot \partial_s \partial_s \dot{\eta}| \leq |\dot{\eta}| \int_0^1 \tilde{C}' \big(1 + |\mathfrak{v}| + |\mathfrak{v}|^2 + |\dot{\eta}|\big) dt.$$

Using Jensen's inequality for integrals and Young's inequality again, we find constants $C', D' > 0$ such that

$$\dot{\eta} \cdot \partial_s \partial_s \dot{\eta} \geq -D' - C' \left(\left| \begin{pmatrix} \mathfrak{v} \\ \dot{\eta} \end{pmatrix} \right|^4 + \int_0^1 \left| \begin{pmatrix} \mathfrak{v} \\ \dot{\eta} \end{pmatrix} \right|^4 dt \right). \qquad (***)$$

Combining $(**)$ and $(***)$ and using the fact that $\left|\left(\begin{smallmatrix} \mathfrak{v} \\ \dot{\eta} \end{smallmatrix}\right)\right|^2 = \left|\partial_s \left(\begin{smallmatrix} v \\ \eta \end{smallmatrix}\right)\right|^2 = w$, we find constants $A, B > 0$ such that

$$\Delta w = |\nabla_{\mathfrak{v}} \mathfrak{v}|^2 + (\partial_s \dot{\eta})^2 + |\nabla_{\dot{v}} \mathfrak{v}|^2 + \kappa \geq -A - B\big(w^2 + \textstyle\int_0^1 w^2 \, dt\big).$$

□

In the remainder of this subsection, we prove the second part of Theorem 25, namely that $|\partial_s(v, \eta)(s,t)|$ converges exponentially to 0 and that (v, η) converges exponentially to some $(v^\pm, \eta^\pm) \in crit\big(\mathcal{A}^H\big)$. We start by describing the structure of the manifold V near points $v^\pm \in \mathcal{N}^{\eta^\pm}$ more explicitly.

Lemma 29 (cf. [6], Lem. A.1)**.** *Assume that $\mathcal{A}^H$ satisfies (MB). Let $\mathcal{N}^\eta \subset \Sigma \subset V$ denote the submanifold of V which is covered by η-periodic orbits of the Reeb field R. Let v be a non-constant η-periodic Reeb trajectory. Then*

a) if $\eta \neq 0$ is the minimal period of v, there exists a tubular neighborhood $U \subset V$ of $im(v)$ such that $U \cap \mathcal{N}^\eta$ is invariant under the flow of ηR and one finds coordinates

$$\big(\vartheta, z_1, \ldots, z_k, z_{k+1}, \ldots, z_{2n-1}\big) \in S^1 \times \mathbb{R}^{2n-1}, \quad k = \dim \mathcal{N}^\eta - 1$$

$$\textit{such that} \quad U \cap \mathcal{N}^\eta = \{z_{k+1} = \cdots = z_{2n-1} = 0\}, \qquad U \cap \Sigma = \{z_{2n-1} = 0\}$$

$$\textit{and} \quad \eta R|_{\mathcal{N}^\eta} = \frac{\partial}{\partial \vartheta}, \qquad Y_\lambda|_U = \frac{\partial}{\partial z_{2n-1}}.$$

b) if v is an m-multiple of a trajectory v_0 of minimal period $\frac{\eta}{m} \neq 0$, there exists a tubular neighborhood $\tilde{U}$ of $im(v_0)$ such that its m-fold cover U together with all the structures induced by the covering map $\pi : U \to \tilde{U}$ from corresponding objects on $\tilde{U}$ satisfies the properties of part a).

Remark. If $\eta = 0$, i.e. if v is constant and $\mathcal{N}^0 = \Sigma$, we can trivially find such coordinates, but there is no distinct direction ϑ and thus $\vartheta = z_0 \in \mathbb{R}$ instead of S^1.

Proof: Let $N_X Y$ denote the normal bundle of a submanifold $Y \subset X$.

- As $\Sigma \subset V$ is a compact codimension 1 submanifold and the Liouville vector field Y_λ transverse to Σ, we can find a tubular neighborhood of Σ given by an embedding
$$\pi_V : \Sigma \times (-\varepsilon, \varepsilon) \hookrightarrow V, \qquad (p, z_{2n-1}) \mapsto \pi_V(p, z_{2n-1})$$
with $\pi_*(\partial_{z_{2n-1}}) = Y_\lambda$ and $\pi(\Sigma \times \{0\}) = \Sigma$. Note that Y_λ trivializes $N_V\Sigma \cong \Sigma \times \mathbb{R}$.

- As $\mathcal{N}^\eta \subset \Sigma$ is a compact submanifold, we can find a tubular neighborhood
$$\tilde{U} \subset N_\Sigma \mathcal{N}^\eta, \qquad \pi_\Sigma : \tilde{U} \hookrightarrow \Sigma, \qquad (q, z_{k+1}, \dots, z_{2n-2}) \mapsto \pi_\Sigma(q, z_{k+1}, \dots, z_{2n-2}),$$
such that $\pi_\Sigma(\mathcal{N}^\eta \times \{0\}) = \mathcal{N}^\eta$. Here, $z_{k+1}, \dots, z_{2n-2}$ are coordinates in the normal direction, which are well-defined only locally.

- In case a), the compact Lie group S^1 acts freely on $\mathcal{N}^\eta$ near v by the flow of ηR. Hence there exists by the Slice Theorem (see [11], Thm. 23.5) a tubular neighborhood of $im(v)$
$$\pi_\mathcal{N} : S^1 \times (-\varepsilon, \varepsilon)^k \hookrightarrow \mathcal{N}^\eta, \qquad (\vartheta, z_1, \dots, z_k) \mapsto \pi_\mathcal{N}(\vartheta, z_1, \dots, z_k),$$
such that $\pi_\mathcal{N}(S^1 \times \{0\}) = im(v)$ and $(\pi_\mathcal{N})_*(\partial_\vartheta) = R$. Note that $N_{\mathcal{N}^\eta}\big(im(v)\big)$ is a trivial bundle over S^1 as it is orientable.

- Combining π_V, π_Σ and $\pi_\mathcal{N}$ then gives the desired tubular neighborhood U and the coordinates. Note that $N_\Sigma\mathcal{N}^\eta$ is trivial over $im(\pi_\mathcal{N})$, as the latter is homotopy equivalent to S^1. This ensures that the coordinates $z_{k+1}, \dots, z_{2n-2}$ are globally defined on U.

- In case b), construct a tubular neighborhood U of v_0 as in case a). Taking its m-fold covering then clearly satisfies the lemma. $\square$

For the following, we choose a finite cover of $\mathcal{N}^{\eta^\pm}$ by neighborhoods U_j as in Lemma 29. We remark that for the asymptotic estimates in $U_j \cap \mathcal{N}^{\eta^\pm}$ it is irrelevant whether we are in a neighborhood of $im(v^\pm, \eta^\pm)$ or on a covering. Let us abbreviate
$$z_{in} := (\vartheta, z_1, \dots, z_k), \qquad z_{out} := (z_{k+1}, \dots, z_{2n-1}), \qquad z := (\vartheta, z_1, \dots, z_{2n-1})$$
and let $n \in \mathbb{R}$ be a coordinate for the η-direction such that $n = 0$ corresponds to $\eta^\pm$. Using the coordinates from Lemma 29, we can express the Rabinowitz-Floer equation (3) near $(v^\pm, \eta^\pm)$ as
$$\begin{aligned} \partial_s Z + J_t(v,\eta)\left(\partial_t Z + \frac{\partial}{\partial\vartheta} - (N + \eta^\pm)\cdot X_H(v)\right) &= 0 \\ \partial_s N + \int_0^1 H(v)dt &= 0, \end{aligned} \tag{14}$$
where $Z(s,t) := \big(\vartheta \circ v(s,t) - t, z \circ v(s,t)\big)$ and $N(s) := \eta(s) - \eta^\pm$ are chosen such that $(Z, N) \to (0,0)$ (which will become clear at the end of this paragraph). Let Z_{in}, Z_{out} resp. Z_{2n-1} denote the z_{in}- resp. z_{out}- resp. z_{2n-1}-part of Z.

As $\eta^{\pm} X_H = \frac{\partial}{\partial \vartheta}$ on $\{z_{out} = 0\}$ and $H \equiv 0$ on $\{z_{2n-1} = 0\}$ we will see in the Lemma below that equation (14) can be rewritten as

$$\begin{aligned} \partial_s Z + J_t(v,\eta)\left[\partial_t Z + S(v,\eta) \cdot \begin{pmatrix} Z_{out} \\ N \end{pmatrix}\right] &= 0 \\ \partial_s N + \int_0^1 \big(h(v) \cdot Z_{2n-1}\big)\, dt &= 0 \end{aligned} \tag{15}$$

for some functions S and h on $S^1 \times \mathbb{R}^{2n}$ resp. $S^1 \times \mathbb{R}^{2n-1}$ with values in $2n \times (2n-k)$-matrices resp. $\mathbb{R}$. Using S and h, we define an s-dependent operator $A(s)$ by

$$A(s) : W^{k+1,p}(S^1, \mathbb{R}^{2n}) \times \mathbb{R} \to W^{k,p}(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$$

$$A(s) \begin{pmatrix} z_{in}(t) \\ z_{out}(t) \\ n \end{pmatrix} = \begin{pmatrix} J_t((v,\eta)(s)) \left[\dfrac{d}{dt} \begin{pmatrix} z_{in}(t) \\ z_{out}(t) \\ n \end{pmatrix} + S\big((v,\eta)(s)\big) \begin{pmatrix} z_{out}(t) \\ n \end{pmatrix} \right] \\ \int_0^1 \big(h(v(s)) \cdot z_{2n-1}(t)\big) dt \end{pmatrix}. \tag{16}$$

Note that we can write more precisely $A(s) = A\big((Z,N)(s,t)\big)$ as we have actually a family of operators depending on t and points in $V \times \mathbb{R}$.

Lemma 30. *Equation (15) holds true. Moreover, for (Z,N) holds that*

$$\Big|\Big| A(s) + \nabla^2 \mathcal{A}^H_{(Z_{in},0,0)} \Big|\Big| \to 0 \qquad \textit{as } s \to \pm\infty.$$

Remark. The operators $A(s)$ are of course only defined in U_j. The last statement is hence to be understood as follows: No matter in which U_j the image of $(v,\eta)(s)$ lies, the operators $A(s)$ become close to the operator $-\nabla^2 \mathcal{A}^H$.

Proof: Equation (15) follows basically from Haddamard's Lemma, which states for a function $f : \mathbb{R}^n \to \mathbb{R}$ with $f|_{\mathbb{R}^k \times \{0\}} \equiv 0$ that

$$f(x) = \sum_{j=k+1}^{n} g_j(x) \cdot x_j, \quad \text{where} \quad g_j(x) := \int_0^1 \frac{\partial f}{\partial x_j}\big(x_1, \dots, x_k, r x_{k+1}, \dots, r x_n\big)\, dr.$$

$$\text{In particular, note that} \quad g_j(x_1, \dots, x_k, 0) = \frac{\partial f}{\partial x_j}\big(x_1, \dots, x_k, 0\big). \tag{$*$}$$

The role of f for S is played by the vector valued function $\mathbb{S} := \frac{\partial}{\partial \vartheta} - (n + \eta^{\pm}) \cdot X_H$ and for h by the function H. The functions S and h are then explicitly given by

- $h(z) := \displaystyle\int_0^1 \frac{\partial H}{\partial z_{2n-1}}\Big(\vartheta, z_1, \dots, z_{2n-2}, r \cdot z_{2n-1}\Big) dr,$
  where only the z_{2n-1}-derivative contributes, as $H \equiv 0$ on $\{z_{2n-1} = 0\}$.

- $S_{ij}(z,n) := \displaystyle\int_0^1 \frac{\partial \mathbb{S}^i}{\partial z_{k+j}}\Big(z_{in}, r z_{out}, r n\Big) dr = -\int_0^1 \frac{\partial\, (n+\eta^{\pm})\big(X_H\big)^i}{\partial z_{k+j}}\Big(z_{in}, r z_{out}, r n\Big) dr$
  where $\big(X_H\big)^i$ is the i-th component of X_H and $\partial z_{2n} = \partial n$. Note again that only the last $2n-k$ derivatives contribute, as $\mathbb{S} \equiv 0$ on $\{z_{out} = 0, n = 0\}$.

We prove the formula only for h, as the proof for S is entirely similar. Let $z \in S^1 \times \mathbb{R}^{2n-1}$ be arbitrary. As $H \equiv 0$ on $\{z_{2n-1} = 0\}$, we have $H(\vartheta, z_1, \dots, z_{2n-2}, 0) = 0$. Let $g(r) := H(\vartheta, z_1, \dots, z_{2n-2}, r \cdot z_{2n-1})$. Then we have

$$\begin{aligned} g'(r) &= \frac{\partial H}{\partial z_{2n-1}}\Big(\vartheta, z_1, \dots, z_{2n-2}, r \cdot z_{2n-1}\Big) \cdot z_{2n-1} \\ \Rightarrow \quad H(z) = g(1) - g(0) = \int_0^1 g'(r)dr &= \int_0^1 \frac{\partial H}{\partial z_{2n-1}}\Big(\vartheta, z_1, \dots, z_{2n-2}, r \cdot z_{2n-1}\Big)dr \cdot z_{2n-1} \\ &= h(z) \cdot z_{2n-1}. \end{aligned}$$

For the asymptotic behaviour of $A(s)$, we first note that on $\{z_{out} = 0, n = 0\}$ we have due to $(*)$ that h and S are given by

$$\begin{aligned} h(z_{in}, 0) &= \frac{\partial H}{\partial z_{2n-1}}\big(z_{in}, 0\big) \\ S_{ij}(z_{in}, 0, 0) &= -\frac{\partial\,(n + \eta^{\pm})\big(X_H\big)^i}{\partial z_{k+j}}\Big(z_{in}, 0, 0\Big). \end{aligned}$$

On the other hand, we recall that $\nabla^2 \mathcal{A}^H_{(v,\eta)}(\mathfrak{v}, \hat{\eta})$ at a critical point $(v, \eta) \in crit\big(\mathcal{A}^H\big)$ was defined by taking a differentiable 1-parameter family (v_a, η_a) with $(v_0, \eta_0) = (v, \eta)$ and $\frac{d}{da}(v_a, \eta_a)\big|_{a=0} = (\mathfrak{v}, \hat{\eta})$ and setting $\nabla^2 \mathcal{A}^H_{(v,\eta)}(\mathfrak{v}, \hat{\eta}) := \frac{d}{da}\nabla \mathcal{A}^H(v_a, \eta_a)\big|_{a=0}$.
Recall that $\nabla \mathcal{A}^H(v_a, \eta_a) = -\big(J(\dot{v}_a - \eta_a X(v_a))\,;\, \int_0^1 H(v_a)dt\big)$. In our local coordinates, any map of the form $(v_0, \eta_0) : S^1 \to S^1 \times \mathbb{R}^{2n},\ t \mapsto (t, z_1, \dots, z_k, 0, \dots, 0, 0)$ for $z_1, \dots, z_k$ fixed corresponds to an element in $crit\big(\mathcal{A}^H\big)$. So if we take an arbitrary family (v_a, η_a) with $(v_0, \eta_0) = (z_{in}, 0, 0)$ and $\frac{d}{da}(v_a, \eta_a)\big|_{a=0} = (\mathfrak{v}, \hat{\eta})$, we get

$$\begin{aligned} \nabla^2 \mathcal{A}^H_{(z_{in},0,0)}(\mathfrak{v}, \hat{\eta}) &= -\frac{d}{da}\begin{pmatrix} J\big(\frac{d}{dt}v_a(t) - (\eta_a + \eta^{\pm})X_H(v_a(t))\big) \\ \int_0^1 H(v_a(t))dt \end{pmatrix}\Bigg|_{a=0} \\ &= -\begin{pmatrix} J\big(\frac{d}{dt}\mathfrak{v}(t) - \sum_{j=0}^{2n} \frac{\partial (n+\eta^{\pm} X_H)^i}{\partial z_j}(z_{in}, 0, 0)[\mathfrak{v}^j]\big) \\ \int_0^1 \sum_{j=0}^{2n-1} \frac{H}{\partial z_j}(z_{in}, 0)[\mathfrak{v}^j]dt \end{pmatrix} \\ &= -\begin{pmatrix} J\big(\frac{d}{dt}\mathfrak{v}(t) - \sum_{j=k+1}^{2n} \frac{\partial (n+\eta^{\pm} X_H)^i}{\partial z_j}(z_{in}, 0, 0)[\mathfrak{v}^j]\big) \\ \int_0^1 \frac{H}{\partial z_{2n-1}}(z_{in}, 0) \cdot \mathfrak{v}^{2n-1}dt \end{pmatrix}, \end{aligned}$$

where the last line follows as $(n + \eta^{\pm})X_H = \frac{\partial}{\partial \vartheta}$ is constant on $\{z_{out} = 0, n = 0\}$ and as H is constant on $\{z_{2n-1} = 0\}$. Note that no derivatives of J appear as the corresponding term is zero due to $\frac{d}{dt}v_0(t) - (0 + \eta^{\pm})X_H(v_0) = 0$ for $(v_0, \eta^{\pm}) \in crit\big(\mathcal{A}^H\big)$.
These calculations show that on $\{z_{out} = 0, n = 0\}$ the operator $-\nabla^2 \mathcal{A}^H_{(z_{in},0,0)}$ has the same form as the point depending operator

$$A_{(z_{in},0,0)}\begin{pmatrix} X_{in} \\ X_{out} \\ X_n \end{pmatrix} := \begin{pmatrix} J\big(\frac{d}{dt}X + S(z_{in}, 0, 0)\begin{pmatrix} X_{out} \\ X_n \end{pmatrix}\big) \\ \int_0^1 h(z_{in}, 0) \cdot X_{2n-1}dt \end{pmatrix}.$$

Recall that Proposition 26 implies for a solution (Z, N) of (14) that $(Z_{out}, N) \to (0, 0)$ uniformly in t as $s \to \pm\infty$. This shows

$$\big|\big|A(s) + \nabla^2 \mathcal{A}^H_{(Z_{in},0,0)}\big|\big| = \big|\big|A_{(Z_{in},Z_{out},N)} - \big(-\nabla^2 \mathcal{A}^H_{(Z_{in},0,0)}\big)\big|\big| \to 0 \qquad \text{uniformly in } t. \ \square$$

Let us write for the moment $\nabla^2\mathcal{A}^H(z_{in}) := \nabla^2\mathcal{A}^H_{(z_{in},0,0)}$. We know by Lemma 20 that the kernels of these operators have finite dimension $k+1 = \dim \mathcal{N}^{\eta^\pm}$. Note that the coordinates z_{in} are by Lemma 29 coordinates on $\mathcal{N}^{\eta^\pm}$ and invariant under the flow of $\eta^\pm R$. We hence obtain that $\ker \nabla^2\mathcal{A}^H(z_{in})$ is for any z_{in} spanned by the following constant S^1-families of vectors corresponding to vectors spanning $T\mathcal{N}^{\eta^\pm}$:

$$e_0(t) = (1,0,\dots,0), \quad e_1(t) = (0,1,0,\dots,0), \quad \dots, \quad e_k(t) = (\underbrace{0,\dots,0}_{k-times},1,0,\dots,0),$$

Recall that we have by Lemma 22 an orthogonal splitting $H^k \times \mathbb{R} = \ker \nabla^2\mathcal{A}^H(z_{in}) \oplus im\,\nabla^2\mathcal{A}^H(z_{in})$. We denote by Q the orthogonal projection onto $im\,\nabla^2\mathcal{A}^H(z_{in}) = \big(\ker \nabla^2\mathcal{A}^H(z_{in})\big)^\perp$. Note that Q does not depend on z_{in}, as $H^k \times \mathbb{R}$ and $\ker \nabla^2\mathcal{A}^H(z_{in})$ do not depend on z_{in}. Moreover, $\nabla^2\mathcal{A}^H(z_{in})$ restricted to $im(Q) = im\,\nabla^2\mathcal{A}^H(z_{in})$ is continuously invertible and we have the formulas

- $\partial_t Q = \partial_t$, as $\ker Q = \ker \nabla^2\mathcal{A}^H(z_{in})$ consists of constant vectors,
- $A(s) = A(s)Q$, as $A(s)|_{\ker \nabla^2\mathcal{A}^H(z_{in})} \equiv 0$ (see (16)),
- $\big(\partial_t A(s)\big) = \big(\partial_t A(s)\big)Q$, as for all vector fields X we have
$$\partial_t(A(s)X) = \partial_t(A(s)QX) \Rightarrow \big(\partial_t A(s)\big)X + A(s)\partial_t X = \big(\partial_t A(s)\big)QX + A(s)\underbrace{\partial_t QX}_{=\partial_t X}$$
$$\Rightarrow \big(\partial_t A(s)\big)X = \big(\partial_t A(s)\big)QX,$$
- $\partial_s Q = Q\partial_s$, as Q is s-independent (as Q does not depend on Z_{in}),
- $\big(\partial_s A(s)\big) = \big(\partial_s A(s)\big)Q$, as for all vector fields X we have
$$\begin{aligned}\partial_s(A(s)X) = \partial_s(A(s)QX) \Rightarrow (\partial_s A(s))X + A(s)\partial_s X &= (\partial_s A(s))QX + A(s)\partial_s QX \\ &= (\partial_s A(s))QX + \underbrace{A(s)Q}_{=A(s)}\partial_s X\end{aligned}$$
$$\Rightarrow (\partial_s A(s))X = (\partial_s A(s))QX,$$
- $\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) + A(s)\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) = 0$, by the Rabinowitz-Floer equation (15),
- $Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s) \to 0$ for $s \to \pm\infty$, as $(Z_{out}, N) \to 0$ and $\partial_t v \to \eta X_H$, which implies in particular that Z_{in} becomes close to being constant with respect to the flow of ηX_H. In our coordinates, this means that Z becomes close to maps of the form
$$t \mapsto (t, z_1, \dots, z_k, 0, \dots, 0), \qquad z_1, \dots, z_k \text{ fixed},$$
which lie all in the kernel of Q.

The following Proposition 32 will prove the remaining part of Theorem 25. In order to give the statement precisely, let us make the following definition.

Definition 31. *Fix a smooth cut-off function β such that $\beta(s) = 1$ for $s \geq 1$ and $\beta(s) = -1$ for $s \leq -1$ and define for $\delta > 0$ the* ***exponential weight function*** *γ_δ by*

$$\gamma_\delta : \mathbb{R} \to \mathbb{R}, \qquad \gamma_\delta(s) = e^{\delta \cdot \beta(s) s}.$$

Let $I \subset \mathbb{R}$ be an unbounded interval. For $\Omega = I$ or $\Omega = I \times S^1$ let $||\cdot||_{k,p,\delta}$ be the following norm for locally p-integrable functions $f : \Omega \to \mathbb{R}$ with weak derivatives up to order k

$$||f||_{k,p,\delta} := \sum_{|i|=0}^{k} ||\gamma_\delta \cdot \partial_i f||_p,$$

where i denotes a multi-index of the (possibly) two variables s and t. The ***γ_δ-weighted Sobolev space*** *$W^{k,p}_\delta(\Omega)$ is then defined by*

$$W^{k,p}_\delta(\Omega) := \left\{ f \in W^{k,p}(\Omega) \,\middle|\, ||f||_{k,p,\delta} < \infty \right\} = \left\{ f \in W^{k,p}(\Omega) \,\middle|\, \gamma_\delta \cdot f \in W^{k,p}(\Omega) \right\}.$$

For $k = 0$, we also write $L^p_\delta(\Omega) := W^{0,p}_\delta(\Omega)$.

Proposition 32 (cf. [7] A.1)**.** *Let $\eta^\pm$ be fixed. There exist constants $C, \rho > 0$ such that for all $\mathcal{A}^H$-gradient trajectories (v, η) with $\lim\limits_{s \to \pm\infty} \partial_s(v, \eta) = 0$ and dist.$\big((v,\eta)(s), \mathcal{N}^{\eta^\pm}\big) \to 0$ holds*

$$\left| \partial_s \begin{pmatrix} v \\ \eta \end{pmatrix} (s,t) \right| = \left| \partial_s \begin{pmatrix} Z \\ N \end{pmatrix} (s,t) \right| \leq C \cdot e^{-\rho |s|}$$

for $|s| \geq s_0 > 0$ sufficiently large. This implies in particular that (v, η) actually converges to some $(v^\pm, \eta^\pm) \in crit\big(\mathcal{A}^H\big)$ and for coordinates (ϑ, z) around $v^\pm$ as in Lemma 29 holds

$$\begin{aligned} \vartheta \circ v(s,t) - t &\in W^{1,p}_\delta\big((-\infty, -s_0] \times S^1, \mathbb{R}\big) \\ z \circ v(s,t) &\in W^{1,p}_\delta\big((-\infty, -s_0] \times S^1, \mathbb{R}^{2n-1}\big) \\ \eta - \eta^- &\in W^{1,p}_\delta\big((-\infty, -s_0], \mathbb{R}\big) \end{aligned}$$

$$\begin{aligned} \vartheta \circ v(s,t) - t &\in W^{1,p}_\delta\big([s_0, \infty) \times S^1, \mathbb{R}\big) \\ z \circ v(s,t) &\in W^{1,p}_\delta\big([s_0, \infty) \times S^1, \mathbb{R}^{2n-1}\big) \\ \eta - \eta^+ &\in W^{1,p}_\delta\big([s_0, \infty), \mathbb{R}\big) \end{aligned}$$

for some s_0 sufficiently large and $\delta/p < \rho$.

Proof: It suffices to make the proof for $s \to +\infty$, the other case being entirely similar. Write as above $Z(s,t) = \big(\vartheta \circ v(s,t) - t, z \circ v(s,t)\big)$ and $N(s) = \eta(s) - \eta^+$ for coordinates (ϑ, z) on some open sets U_j covering $\mathcal{N}^{\eta^+}$ as in Lemma 29. We will show below, that there exist constants $\rho, \tilde{C} > 0$ such that for $s \geq s_0$ sufficiently large and on any U_j holds

$$||Q \left(\tfrac{Z}{N} \right) (s)||_1 \leq \tilde{C} \cdot e^{-\rho s} \qquad (*)$$

for the H^1-norm $||\cdot||_1$. As $0 = \partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) + A(s) \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) = \partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) + A(s) Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)$, this yields

$$||\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s)||_0^2 \leq ||A(s)||^2 \cdot ||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s)||_1^2 \leq \tilde{C} \cdot e^{-2\rho s},$$

for a bigger constant $\tilde{C}$ and $s \geq s_0$. As (Z, N) satisfies the partial differential equation (14), which is similar to (3), it satisfies also the mean value inequality from Lemma 28, i.e. there exist constants $A, \delta > 0$ such that

$$\int_{B_r(s,t)} |\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)|^2 < \delta \quad \forall t \in S^1$$

$$\text{implies} \qquad \exists\, t^* \in S^1 \ : \ |\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s,t)|^2 \leq \frac{Ar^2}{2} + \frac{8}{\pi r^2} \int_{B_r(s,t^*)} |\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)|^2 \quad \forall t \in S^1.$$

If necessary, increase s_0 such that $\tilde{C} \cdot e^{-2\rho(s_0-1)} \leq \delta$ holds and set $r = e^{-\rho s/2}$ for $s \geq s_0$. Then $s - r \geq s - 1 \geq s_0 - 1$ so that the assumption of the mean value inequality is satisfied and we get

$$\begin{aligned} & |\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s,t)|^2 \leq \frac{A}{2} e^{-\rho s} + \frac{8\tilde{C}}{\pi e^{-\rho s}} \cdot e^{-2\rho(s-1)} = \left(\frac{A}{2} + \frac{8\tilde{C}e^{2\rho}}{\pi}\right) e^{-\rho s} \\ \Rightarrow \qquad & |\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s,t)| \leq \tilde{C} \cdot e^{-\rho s}, \end{aligned}$$

for $\tilde{C}$ even bigger. Now, we can show that (v, η) truly converges. For any interval $[s_1, s_2]$ such that $(v, \eta)\big([s_1, s_2]\big) \subset U_j$ for a fixed j, we obtain by integration

$$\begin{aligned} \left| \begin{pmatrix} Z \\ N \end{pmatrix} (s_2, t) - \begin{pmatrix} Z \\ N \end{pmatrix} (s_1, t) \right| = \left| \int_{s_1}^{s_2} \partial_{\mathfrak{s}} \begin{pmatrix} Z \\ N \end{pmatrix} (\mathfrak{s}, t)\, d\mathfrak{s} \right| &\leq \int_{s_1}^{s_2} \tilde{C} \cdot e^{-\rho \mathfrak{s}}\, d\mathfrak{s} \\ &= \tilde{C} \cdot \frac{1}{\rho} \cdot e^{-\rho s_1} \big(1 - e^{-\rho(s_2 - s_1)}\big) \\ &\leq \tilde{C} \cdot \frac{1}{\rho} \cdot e^{-\rho s_1}. \end{aligned}$$

As the U_j form a finite open cover of the compact set $\mathcal{N}^{\eta^+}$, we find that the maximal distance $dist.(p, \partial U_j)$ over all U_j for any $p \in \mathcal{N}^{\eta^+}$ is bounded from below. So for s_1 large enough, we know from the above estimate that $(v, \eta)(s)$ stays in one U_j for all $s \geq s_1$. Moreover, $(v, \eta)(s)$ is a Cauchy sequence, which implies that it converges to some $(v^+, \eta^+) \in U_j \cap \mathcal{N}^{\eta^+}$. Taking once more with Lemma 29 a neighborhood U around (v^+, η^+) such that in these coordinates $\lim \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) = 0$, we finally obtain

$$\left| \begin{pmatrix} Z \\ N \end{pmatrix} (s, t) \right| = \left| \int_s^\infty \partial_{\mathfrak{s}} \begin{pmatrix} Z \\ N \end{pmatrix} (\mathfrak{s}, t)\, d\mathfrak{s} \right| \leq \int_s^\infty \tilde{C} \cdot e^{-\rho \mathfrak{s}}\, d\mathfrak{s} = \tilde{C} \cdot \frac{1}{\rho} \cdot e^{-\rho s},$$

which proves the proposition with $C := \tilde{C}/\rho$.

It remains to show the exponential H^1-estimate $(*)$ on $Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)$. In the following, we abbreviate $||\cdot|| := ||\cdot||_0$ for the L^2-norm, write $||\cdot||_k$ for the H^k-norm and $\langle \cdot, \cdot \rangle = g(\cdot,\cdot)$ for the scalar product on L^2. We first show that $||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||$ is exponentially bounded. For that define

$$f(s) := \frac{1}{2}\left|\left|Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s)\right|\right|^2.$$

We show below the existence of a constant c such that f'' satisfies for s sufficiently large

$$f''(s) \geq 4(c - 2\varepsilon) \cdot f(s), \tag{$**$}$$

where $\varepsilon > 0$ is an arbitrarily small constant (for s sufficiently large). Set $\rho := \sqrt{c - 2\varepsilon}$, such that $f'' \geq 4\rho^2 \cdot f$. For s_0 large, define furthermore $g(s) := f(s_0)e^{-2\rho(s-s_0)}$. Then

$$g'' = 4\rho^2 g, \qquad (f-g)'' \geq 4\rho^2(f-g), \qquad (f-g)(s_0) = 0 \quad \text{and} \quad \lim_{s\to\infty} f(s) - g(s) = 0.$$

The last statement holds as $g(s) \to 0$ and $f(s) \to 0$. To see this for f, recall that $Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) \to 0$. Then it follows that $f - g \leq 0$ on $[s_0, \infty)$, as it cannot have a strictly positive maximum. Therefore we obtain an exponential bound as

$$||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s)|| = f(s) \leq g(s) = ||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s_0)||\, e^{-\rho(s-s_0)}.$$

To show $(**)$, consider the operator $A(s)$ and recall that $||A(s) + \nabla^2\mathcal{A}^H(Z_{in})|| \to 0$. As all $\nabla^2\mathcal{A}^H(Z_{in})$ restricted to $im(Q)$ are continuously invertible, we find for s sufficiently large that the operators $A(s)$ and $QA(s)$ are also invertible when restricted to $im(Q)$. Hence, there exists for such s constants $c(s) > 0$ such that for all $\left(\begin{smallmatrix} z \\ n \end{smallmatrix}\right) \in H^k(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ holds

$$\left|\left|A(s)Q\left(\begin{smallmatrix} z \\ n \end{smallmatrix}\right)\right|\right|_{k-1}^2 \geq \left|\left|QA(s)Q\left(\begin{smallmatrix} z \\ n \end{smallmatrix}\right)\right|\right|_{k-1}^2 \geq c(s) \cdot \left|\left|Q\left(\begin{smallmatrix} z \\ n \end{smallmatrix}\right)\right|\right|_{k}^2 \geq c(s) \cdot \left|\left|Q\left(\begin{smallmatrix} z \\ n \end{smallmatrix}\right)\right|\right|_{k-1}^2.$$

Note that $c(s)$ can be chosen arbitrarily close to the smallest non-zero square of an eigenvalue of $\nabla^2\mathcal{A}^H(Z_{in})$ (while increasing s). As $\nabla^2\mathcal{A}^H(z_{in})$ depends continuously on $z_{in} \in \mathcal{N}^{\eta^+}$ and $\dim\left(\ker \nabla^2\mathcal{A}^H(z_{in})\right) = \dim \mathcal{N}^{\eta^+}$ is constant on the compact set $\mathcal{N}^{\eta^+}$, we can choose one c independent of s for which the above estimate holds for all s sufficiently large. Note that c can in particular be chosen independent from U_j.
The estimate $(**)$ now follows from simple but tedious estimates. The main idea is to replace $\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)$ by $-A(s)\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)$ via the Rabinowitz-Floer equation and then to apply the above estimate. In order to make the calculations more readable, we will from now on write A instead of $A(s)$. Using the formulas for Q (see page 36), we calculate for f

$$\begin{aligned}
f''(s) &= ||\partial_s Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||^2 + \langle Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), \partial_s^2 Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\rangle + \mu \\
&= ||QAQ\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||^2 - \langle Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), \partial_s QA\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\rangle + \mu \\
&= ||QAQ\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||^2 - \langle Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), Q(\partial_s A)Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) - QA^2Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\rangle + \mu \\
&= ||QAQ\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||^2 - \langle Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), Q(\partial_s A)Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\rangle \\
&\qquad\qquad + \langle (A^* - A)Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), AQ\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\rangle + ||AQ\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||^2 + \mu \\
&\geq 2c||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 - ||\partial_s A|| \cdot ||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 - ||A^* - A|| \cdot ||A|| \cdot ||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 + \mu \\
&\geq 2(c - \varepsilon)||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 + \mu.
\end{aligned}$$

Here, A^* denotes the adjoint of A and we used the facts that $||A^* - A|| \to 0$ (as the limit operators $\nabla^2 \mathcal{A}^H(Z_{in})$ are selfadjoint), that $||\partial_s A|| \to 0$ (as $\partial_s A \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) = (DA)[\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)]$ and $\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) \to 0$) and that $||A||$ is uniformly bounded.
With μ we summarized all terms of f'' which involve derivatives of the metric g. Note that we cannot avoid this, as J and hence g depends on the point $\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)(s)$. The estimate of μ is similar and goes as follows:

$$\begin{aligned}
\mu &= \tfrac{1}{2}\big(\partial_s^2 g\big)\big(Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\big) + \big(\partial_s g\big)\big(Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), \partial_s Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\big) \\
&= \tfrac{1}{2}\big(\partial_s(Dg)[\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)]\big)\big(Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\big) + \big(Dg[\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)]\big)\big(Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), QAQ\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\big) \\
&= \ \tfrac{1}{2}\big(D^2 g[\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), \partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)] + Dg[(\partial_s A)Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right) + A^2 Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)]\big)\big(Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\big) \\
&\quad + \big(Dg[\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)]\big)\big(Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right), QAQ\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\big) \\
&\geq -\tfrac{1}{2}||D^2 g|| \cdot ||\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||^2 \cdot ||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 - \tfrac{1}{2}||Dg||\big(||\partial_s A|| + ||A^2||\big)||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1 \cdot ||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 \\
&\quad - ||Dg|| \cdot ||QA|| \cdot ||\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)|| \cdot ||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 \\
&\geq -2\varepsilon ||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2.
\end{aligned}$$

Here, we write Dg for the total differential of g, which is in coordinates well-defined. The last line follows then for s sufficiently large, as $||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1, ||\partial_s\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)|| \to 0$, while all operator norms are uniformly bounded. Combining the two estimates, we obtain

$$f''(s) \geq 2(c - 2\varepsilon)||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||_1^2 \geq 2(c - 2\varepsilon)||Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||^2 = 4(c - 2\varepsilon) f(s).$$

To complete the proof of $(*)$, we also have to show that $||\partial_t Q\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)|| = ||\partial_t\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)||$ is exponentially bounded. We define again a function

$$f(s) := \frac{1}{2}\Big|\Big|\partial_t(s)\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)\Big|\Big|^2$$

and show the existence of another constant $\tilde{c} > 0$ such that f'' satisfies for s sufficiently large

$$f''(s) \geq 4(\tilde{c} - 3\varepsilon) \cdot f(s) - K e^{-2\rho s}, \tag{$***$}$$

where $\varepsilon > 0$ is again arbitrarily small, K is some constant and ρ is as above. By choosing $\tilde{\rho} \leq \min\{\sqrt{\tilde{c} - 3\varepsilon}, \rho\}$, we then get $f'' \geq 4\tilde{\rho}^2 \cdot f - K e^{-2\tilde{\rho} s}$. For s_0 sufficiently large, define

$$g(s) := \left(\frac{sK}{4\tilde{\rho}} - \frac{s_0 K}{4\tilde{\rho}} + f(s_0) e^{2\tilde{\rho} s_0}\right) e^{-2\tilde{\rho} s}.$$

Then we have $g'' = 4\tilde{\rho}^2 g - K e^{-2\tilde{\rho} s}$, $g(s_0) = f(s_0)$ and $\lim\limits_{s \to \infty} g(s) = 0$ and hence

$$(f - g)'' \geq 4\tilde{\rho}^2 (f - g), \quad (f - g)(s_0) = 0 \quad \text{and} \quad \lim_{s \to \infty} f(s) - g(s) = 0.$$

Then it follows again that $f - g \leq 0$ on $[s_0, \infty)$, as it cannot have a strictly positive maximum. Therefore, we obtain

$$||\partial_t\left(\begin{smallmatrix} Z \\ N \end{smallmatrix}\right)|| \leq \left(\frac{sK}{4\tilde{\rho}} - \frac{s_0 K}{4\tilde{\rho}} + f(s_0) e^{2\tilde{\rho} s_0}\right) e^{-2\tilde{\rho} s} \leq \tilde{C}(s + 1) e^{-2\tilde{\rho} s}$$

for some large constant $\tilde{C}$.

As $(s+1)e^{-\delta s} \to 0$ for $s \to \infty$ and any $\delta > 0$, we get by decreasing $\tilde{\rho}$ and increasing s_0 further, that for all $s \geq s_0$ holds

$$||\partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) || \leq \tilde{C} \cdot e^{-\tilde{\rho} s}.$$

To show $(***)$, we consider the operator ∂_t on H^1. Its image $im(\partial_t)$ is closed in L^2, as it has a right inverse by integration. Moreover, $im(\partial_t) \cap \ker \nabla^2 \mathcal{A}^H(z_{in}) = 0$ for all z_{in}, as integrating a constant vector field $x(t) = x_0$ over S^1 yields $X(t) = tx_0$, which is 1-periodic only if $x_0 = 0$. As $\nabla^2 \mathcal{A}^H(z_{in})(im(\partial_t))$ is also closed, we find that $\nabla^2 \mathcal{A}^H(z_{in})$ restricted to $im(\partial_t)$ is a bijective operator between Banach spaces and hence continuously invertible. It follows that $A(s)$ for s sufficiently large is also continuously invertible over $im(\partial_t)$, which gives us for s sufficiently large a constant $\tilde{c}$ such that for all $\left(\begin{smallmatrix} z \\ n \end{smallmatrix} \right) \in H^k(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ holds

$$||A(s)\partial_t \left(\begin{smallmatrix} z \\ n \end{smallmatrix} \right) ||_{k-1}^2 \geq \tilde{c} \cdot ||\partial_t \left(\begin{smallmatrix} z \\ n \end{smallmatrix} \right) ||_{k-1}^2.$$

Note that we can choose $\tilde{c}$ globally, as all $\nabla^2 \mathcal{A}^H(z_{in})$ restricted to $im(\partial_t)$ are injective. For the proof of $(***)$, we use estimates similar to whose for $||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||$

$$\begin{aligned} f''(s) &= ||\partial_s \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2 + \langle \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), \partial_s^2 \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle + \mu \\ &= ||\partial_t \partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2 + \langle \partial_t, \partial_t \partial_s^2 \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle + \mu \\ &= ||\partial_t A \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2 - \langle \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), \partial_t (\partial_s A) \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle + \langle \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), \partial_t A^2 \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle + \mu \\ &= ||A \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2 - \langle \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), (\partial_s A) \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle + \langle \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), A^2 \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle + \mu + \kappa, \end{aligned}$$

where μ contains again all terms with derivatives of the metric and κ is an error term which we give explicitly below. Using similar estimates as for $||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||$, we find that

$$f''(s) \geq 2(\tilde{c} - 2\varepsilon) ||\partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2 + \kappa,$$

with $\varepsilon > 0$ arbitrarily small and s sufficiently large. The error term κ is given by

$$\begin{aligned} \kappa = \quad & ||(\partial_t A) \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2 + 2 \langle A \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), (\partial_t A) \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle \\ & - \langle \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), (\partial_t \partial_s A) \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle + \langle \partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right), (\partial_t A^2) \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) \rangle \\ \geq & - 2 ||A|| \cdot ||\partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) || \cdot ||\partial_t A|| \cdot ||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) || \\ & - ||\partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) || \cdot ||\partial_t \partial_s A|| \cdot ||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) || - ||\partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) || \cdot ||\partial_t A^2|| \cdot ||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) || \\ \geq & -2\varepsilon ||\partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2 - K \cdot ||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||^2. \end{aligned}$$

Here, K is some large constant depending on ε. The last line follows from Young's inequality and the fact that all operator norms are bounded. Using the exponential bound for $||Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||$, we hence get

$$f''(s) \geq 2(\tilde{c} - 3\varepsilon) ||\partial_t \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right) ||_1^2 - K \cdot e^{-2\rho s} \geq 4(\tilde{c} - 3\varepsilon) f(s) - K \cdot e^{-2\rho s}. \qquad \square$$

Remark. By making similar estimates for the higher t-derivatives of $Q \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right)$, we could get an exponential bound on the H^k-norm of $\partial_s \left(\begin{smallmatrix} Z \\ N \end{smallmatrix} \right)$. The pointwise estimates would then follow from Sobolev's embedding theorem.

2.3. Unique continuation and injective points

In this section, we present certain properties of solutions of the Rabinowitz-Floer equation (3) which they have in common with holomorphic curves – in particular unique continuation and the existence of so called regular or injective points. These properties for Floer/symplectic homology were first shown by Floer, Hofer, Salamon in [24]. Recently, Bourgeois, Oancea, [8], and Abbondandolo, Merry, [1], generalized them to Rabinowitz-Floer homology. I extend them here to situations, where we have on V a symplectic symmetry of finite order.

That solutions of (3) can be to some extend considered as holomorphic curves is due to the Carleman similarity principle (see [24], Thm. 2.2, or [36], section 2.3). It states that every sufficiently regular solution u with $u(0) = 0$ of a partial differential equation on the disc $B_\varepsilon(0) \subset \mathbb{C}$ of the form

$$\partial_s u(z) + J(z)\partial_t u(z) + C(z)u(z) = 0$$

is conjugated to a holomorphic curve. Here, J, C are supposed to be $\mathbb{R}$-linear, sufficiently differentiable and $J^2 = -Id$. Note that even solutions of the "normal" Floer equation (54) do not satisfy this equation, as $X_H(p) \neq 0$ in general for any $p \in V$. However, the difference of two solutions u, v of (54) satisfies this equation, provided that u and v agree to infinite order at a point p. It follows then from the unique continuation for holomorphic maps that u and v coincide on an open neighborhood of p and thus everywhere.
Unique continuation can also be obtained from Aronszajn's Theorem (see [36], 2.3), which states that every (sufficiently regular) function u on $B_\varepsilon(0)$ is equal to 0 if it vanishes to infinite order at 0 and satisfies point-wise almost everywhere

$$|\Delta u| \leq c \cdot (|u| + |\partial_s u| + |\partial_t u|)\,.$$

Unfortunately both results, Carleman similarity principle and Aronszajn's Theorem, do not apply to solutions of the Rabinowitz-Floer equation, as it involves an integral and is hence not completely local. However, Bourgeois and Oancea generalized in [8] Aronszajn's Theorem to situations like this and used it then to prove the following two Theorems 33 and 34.

To state the theorems, we write $I_h(s) := (s-h, s+h) \subset \mathbb{R}$ and $V_h(s,t) := (s-h, s+h) \times (t-h, t+h) \subset \mathbb{R} \times S^1$ for $h > 0$.

Theorem 33 (**Unique Continuation**, cf. [8], Prop. 3.5)**.**
Let $v_i : I_h(s) \times S^1 \to V,\ \eta_i : I_h(s) \to \mathbb{R},\ i = 0, 1,$ be two smooth functions satisfying the (s-dependent) Rabinowitz-Floer equation, i.e.

$$\left.\begin{aligned} \partial_s v + J_t(v,\eta)\big(\partial_t v - \eta X_H(v)\big) = 0 \\ \partial_s \eta + \textstyle\int_{S^1} H(v)dt = 0 \end{aligned}\right\}(3)\ \ \textit{or}\qquad \left.\begin{aligned} \partial_s v + J_t(v,\eta)\big(\partial_t v - \eta X_{H_s}(v)\big) = 0 \\ \partial_s \eta + \textstyle\int_{S^1} H_s(v)dt = 0 \end{aligned}\right\}(20).$$

If (v_0, η_0) and (v_1, η_1) coincide at a point $p \in I_h(s) \times S^1$ to infinite order, then they coincide on $I_h(s) \times S^1$. In particular, this applies then (v_0, η_0) and (v_1, η_1) agree on some open set $U \subset I_h(s) \times S^1$.

Theorem 34 (cf. [8], Lem. 4.5, or [24], Lem. 4.2)**.**
Suppose that $U_i = (v_i, \eta_i)$, $i = 0, 1$, are smooth functions on $I_{h_0}(s)\times S^1$, $h_0 > 0$, satisfying the Rabinowitz-Floer equation (3). Assume that

$$U_0(s_0, t_0) = U_1(s_0, t_0), \qquad \partial_s v_0(s_0, t_0) \neq 0, \qquad \partial U_1(s_0, t_0) \neq 0.$$

Assume also that for any $0 < h' \leq h_0$ there exists $0 < h \leq h_0$ such that for any $(s, t) \in V_h(s_0, t_0)$ there exists $(s', t) \in V_{h'}(s_0, t_0)$ such that $U_0(s, t) = U_1(s', t)$. Then $U_0 = U_1$.

For the following generalized results on injective points, let us consider the following situation: Suppose we have on V a smooth (exact) symplectic symmetry σ of finite order, i.e. $\sigma : V \to V$ is a diffeomorphism such that $\sigma^k = Id$ for some $k \in \mathbb{N}$ and $\sigma^*\lambda = \lambda$. Moreover, suppose that H and J are σ-invariant, i.e. $H(\sigma(p)) = H(p)$ for all $p \in V$ and $\sigma^* J = J$. Note that this implies that for any solution (v, η) of (3) we have that $(\sigma \circ v, \eta)$ is also a solution of (3). Let $V_{fix} := \{p \in V \,|\, \sigma(p) = p\}$ denote the fixed point set of σ.

Lemma 35. *Let $U = (v, \eta)$ be a solution of the Rabinowitz-Floer equation (3). Suppose that $im(v) \not\subset V_{fix}$. Then, the following set is open and dense in $\mathbb{R} \times S^1$:*

$$F(U) := \{(s, t) \in \mathbb{R} \times S^1 \,|\, v(s, t) \notin V_{fix}\}.$$

Proof: It is easy to see that $F(U)$ is open in $\mathbb{R} \times S^1$, as we may write it equivalently as

$$F(U) := \big\{(s, t) \in \mathbb{R} \times S^1 \,\big|\, dist.\big(v(s, t), (\sigma \circ v)(s, t)\big) > 0\big\},$$

where the condition $dist.(\cdot, \cdot) > 0$ is obviously open. To show that $F(U)$ is dense, we suppose the contrary. Then there exists an open set $W \subset \mathbb{R}\times S^1$ such that $v(W) \subset V_{fix}$. It follows that (v, η) and $(\sigma \circ v, \eta)$ coincide on W and hence by unique continuation that $(v, \eta) = (\sigma \circ v, \eta)$ everywhere. But this implies that $im(v) \subset V_{fix}$, a contradiction to the premise of the lemma. $\square$

Lemma 36. *Suppose that (v, η) is a solution of (3) with $\lim\limits_{s\to\pm\infty}(v, \eta) = (v^\pm, \eta^\pm) \in crit\big(\mathcal{A}^H\big)$. If $(\partial_s v, \partial_s \eta) \not\equiv (0, 0)$, then there is no constant $s_0 \in \mathbb{R} \setminus \{0\}$ such that (v, η) is an s_0-shift of itself, i.e. it cannot hold for every $(s, t) \in \mathbb{R} \times S^1$ that*

$$\big(v(s + s_0, t), \eta(s + s_0)\big) = \big(v(s, t), \eta(s)\big).$$

In the σ-symmetric case, there is also no constant $s_0 \in \mathbb{R} \setminus \{0\}$ such that $(\sigma \circ v, \eta)$ is an s_0-shift of (v, η). If $im(v) \not\subset V_{fix}$, then $s_0 = 0$ is also impossible.

Proof: If there were such a constant $s_0 \neq 0$, then (v, η) would be s_0-periodic and hence

$$\begin{aligned}
\big(v(s, t), \eta(s)\big) &= \big(v(s + k \cdot s_0, t), \eta(s + k \cdot s_0)\big) \qquad \forall\, k \in \mathbb{Z},\ \forall\, s, t \in \mathbb{R} \times S^1.\\
&= \lim_{k\to\pm\infty} \big(v(s + k \cdot s_0, t), \eta(s + k \cdot s_0)\big)\\
&= \big(v^\pm(t), \eta^\pm\big).
\end{aligned}$$

This implies that $\partial_s(v, \eta) \equiv (0, 0)$ – a contradiction to our assumption.

If there were such a constant s_0 in the σ-symmetric case, then we find by applying σ iteratively k times that (v, η) would be $k \cdot s_0$ periodic. By repeating the same arguments as before, we find again a contradiction. If $im(\sigma) \not\subset V_{fix}$, then we have $\sigma \circ v \neq v$, which shows that $s_0 = 0$ is also impossible. $\square$

Following [24] and [8], we define for a solution $U = (v, \eta)$ of (3) with $\lim_{s \to \pm\infty} (v, \eta) = (v^\pm, \eta^\pm) \in crit(\mathcal{A}^H)$ the set of regular points as

$$\mathcal{R}(U) := \left\{ (s,t) \in \mathbb{R} \times S^1 \;\middle|\; \begin{array}{l} \partial_s\big(v(s,t), \eta(s)\big) \neq (0,0) \\ \quad (v(s,t), \eta(s)) \neq (v^\pm(t), \eta^\pm) \\ \quad (v(s,t), \eta(s)) \neq \big(v(s',t), \eta(s')\big), \forall s' \in \mathbb{R} \setminus \{s\} \end{array} \right\}.$$

Together with Peter Uebele, [50], we define in the σ-symmetric case when $im(v) \not\subset V_{fix}$ the set of symmetric regular points by

$$\mathcal{S}_\sigma(U) := \left\{ (s,t) \in \mathbb{R} \times S^1 \;\middle|\; \begin{array}{l} \partial_s\big(v(s,t), \eta(s)\big) \neq (0,0) \\ \quad (v(s,t), \eta(s)) \neq (\sigma \circ v^\pm(t), \eta^\pm) \\ \quad (v(s,t), \eta(s)) \neq \big(\sigma \circ v(s',t), \eta(s')\big), \forall s' \in \mathbb{R} \end{array} \right\}.$$

Note that we require in particular for $(s,t) \in \mathcal{S}_\sigma(U)$ that $v(s,t) \neq (\sigma \circ v)(s,t)$.

Proposition 37 (cf. [24], Thm. 4.3, or [8], Prop. 4.3)**.**
Assume that $\partial_s U \not\equiv 0$. Then, the set $\mathcal{R}(U)$ is open and dense in the non-empty open set $\{(s,t) \in \mathbb{R} \times S^1 \,|\, \partial_s v(s,t) \neq 0\}$. In the symmetric case, the set $\mathcal{S}_\sigma(U)$ is likewise open and dense in the same set.

Proof:
<u>1) All conditions are open</u>

- The set $\{(s,t) \in \mathbb{R} \times S^1 \,|\, \partial_s v(s,t) \neq 0\}$ is clearly open. If it were empty, then $\partial_s v \equiv 0$ and $v(s, \cdot) = v^\pm(\cdot) \in \Sigma = H^{-1}(0)$ for all s. Then, the second equation of (3) implies that $\partial_s \eta \equiv 0$ and hence $\partial_s U \equiv 0$, which contradicts the assumption of the proposition.

- The first and second condition for $\mathcal{R}(U)$ resp. $\mathcal{S}_\sigma(U)$ are clearly open. We need to show that the third condition for $\mathcal{R}(U)$ is open as well. Arguing by contradiction, we find a point $(s_0, t_0) \in \mathcal{R}(U)$, a sequence $(s^\nu, t^\nu) \to (s_0, t_0)$ and a sequence $s'^\nu \neq s^\nu$ such that $U(s'^\nu, t^\nu) = U(s^\nu, t^\nu)$. As $\partial_s U(s_0, t_0) \neq 0$, we can find $h > 0$ such that $U(\cdot, t_0)$ is an embedding on $I_h(s_0)$ and $U(\cdot, t^\nu)$ is an embedding on $I_h(s^\nu)$ for ν large enough. Thus we can assume without loss of generality that s'^ν is bounded away from s_0 (otherwise $s'^\nu \in I_h(s^\nu)$ for ν large enough, a contradiction). Since U converges at $\pm\infty$ to $(v^\pm, \eta^\pm)$ and $U(s_0, t_0) \neq (v^\pm(t_0), \eta^\pm)$ by assumption, we infer the existence of some $T > 0$ such that $s'^\nu \in [-T, T]$ for all ν. We can therefore extract a convergent subsequence, still denoted by s'^ν, such that $s'^\nu \to s'_0 \neq s_0$. Then $U(s'_0, t_0) = U(s_0, t_0)$, which contradicts the assumption $(s_0, t_0) \in \mathcal{R}(U)$.

- For $\mathcal{S}_\sigma(U)$, the third condition can be written as

$$dist\big(v(s,t),\eta(s)\big),(\sigma\circ v)(\mathbb{R},t),\eta(\mathbb{R})\big)>0,$$

since $(v(s,t),\eta(s))\neq(v^\pm(t),\eta^\pm)$. This is clearly an open condition.

2) Density

It suffices to show for every $(s_0,t_0)\in\mathbb{R}\times S^1$ with $\partial_s v(s_0,t_0)\neq 0$ that it can be approximated by a sequence of points $(s^\nu,t^\nu)\in\mathcal{R}(U)$ resp. $\in\mathcal{S}_\sigma(U)$. As $im(\sigma)\not\subset V_{fix}$, we know by Lemma 35 that any $(s_0,t_0)\in\mathbb{R}\times S^1$ can be approximated by a sequence (s^ν,t^ν) satisfying $v(s^\nu,t^\nu)\notin V_{fix}$. Hence we may assume without loss of generality that $(v_0,t_0)\notin V_{fix}$. Thus, any sequence $(s^\nu,t^\nu)\to(s_0,t_0)$ satisfies $v(s^\nu,t^\nu)\notin V_{fix}$ for ν large enough. Hence, we may assume for $S_\sigma(U)$ that $v(s,t)\neq(\sigma\circ v)(s,t)$. The remaining conditions for $\mathcal{S}_\sigma(U)$ are very similar to those of $\mathcal{R}(U)$ and the proof of the density of $\mathcal{R}(U)$ and $\mathcal{S}_\sigma(U)$ is similar as well. In order to give them both at the same time, we will write $W(s',t)=(w(s',t),\eta(s'))$, where W and w are either U and v or $\sigma\circ U$ and $\sigma\circ v$. As $\partial_s v(s_0,t_0)\neq 0$, we may choose $h>0$ so small that $\partial_s v\neq 0$ on $V_h(s_0,t_0)$ and $I_h(s_0)\to V, s\mapsto v(s,t)$ is an embedding for all $t\in I_h(t_0)$. Then $I_h(s_0)\to V\times\mathbb{R}, s\mapsto U(s,t)$ is a fortiori also an embedding for all $t\in I_h(t_0)$. Thus, every point $(s,t)\in V_h(s_0,t_0)$ can be approximated by a sequence (s^ν,t^ν) satisfying $U(s^\nu,t^\nu)\neq(v^\pm(t^\nu),\eta^\pm)$ and $U(s^\nu,t^\nu)\neq(w^\pm(t^\nu),\eta^\pm)$. Hence we can assume without loss of generality that

$$\forall(s,t)\in V_h(s_0,t_0)\ :\ U(s,t)\neq(v^\pm(t),\eta^\pm),(w^\pm(t),\eta^\pm). \qquad (*)$$

Let us denote $C(W):=\{(s,t)\in\mathbb{R}\times S^1\,|\,\partial_s W(s,t)=0\}$. As $\partial_s U\not\equiv 0$ this set has empty interior by unique continuation.

Claim: *(s_0,t_0) can be approximated by a sequence (s^ν,t_0) such that for all ν and all $s'\in\mathbb{R}\setminus\{s^\nu\}$ with $U(s^\nu,t_0)=W(s',t_0)$ holds that $(s',t_0)\notin C(W)$.*

Assuming the claim, we can suppose without loss of generality that for each $s'\in\mathbb{R}$ with $U(s_0,t_0)=W(s',t_0)$ holds that $(s',t_0)\notin C(W)$. Moreover, after further diminishing $h>0$, we can assume without loss of generality that

$$\forall(s,t)\in V_h(s_0,t_0),\forall s'\in\mathbb{R}\ :\ U(s,t)=W(s',t)\ \Rightarrow\ (s',t)\notin C(W). \qquad (**)$$

Indeed, if this would fail for all $h>0$, we could find a sequence $(s^\nu,t^\nu)\to(s_0,t_0)$ and a sequence s'^ν such that $(s'^\nu,t^\nu)\in C(W)$ and $U(s^\nu,t^\nu)=W(s'^\nu,t^\nu)$. Due to $\lim_{s\to\pm\infty}W(s,t)=(w^\pm(t),\eta^\pm)$ uniformly, we deduce from $(*)$ the existence of a constant $T>0$ such that $|s'^\nu|\leq T$. Thus, up to a subsequence, we have $s'^\nu\to s'\in[-T,T]$, $t^\nu\to t_0$ and $U(s_0,t_0)=W(s',t_0)$ with $(s',t_0)\in C(W)$ contradicting our last assumption on (s_0,t_0) obtained by the claim.

Proof of the claim: Let us choose a neighborhood $\mathcal{V}$ of $U(I_h(s_0),t_0)$ in $V\times\mathbb{R}$ of the form $I_h(s_0)\times\mathbb{R}^{2n}$. This is possible, as $s\mapsto v(s,t_0)$ is an embedding. Let pr_1 denote the projection to the first coordinate. Consider the function $f:=pr_1\circ W(\cdot,t_0)$ with

$$f:dom(f):=W(\cdot,t_0)^{-1}(\mathcal{V})\to I_h(s_0).$$

Write $C(W)_{t_0} := \{s \in \mathbb{R} \,|\, (s,t_0) \in C(W)\}$. Then $f\big(C(W)_{t_0} \cap dom(f)\big)$ is contained in the critical values of f. This is a nowhere dense set in $I_h(s_0)$ by Sard's Theorem and the claim follows.

Now assume by contradiction the existence of a point (s_0,t_0) satisfying $\partial_s v(s_0,t_0) \neq 0$ and $(*)$ and $(**)$, which cannot be approximated by points in $\mathcal{R}(U)$ resp. $\mathcal{S}_\sigma(U)$. Then there exists an $0 < \varepsilon < h$ such that

$$\forall\, (s,t) \in V_\varepsilon(s_0,t_0) \;\; \exists\, s' \neq s \;:\; U(s,t) = W(s',t). \tag{$***$}$$

As above, we find a constant $T > 0$ such that $|s'| \leq T$ for all s'. This implies that for any $(s,t) \in V_\varepsilon(s_0,t_0)$, there is only a finite number of values $s' \in \mathbb{R}$ such that $U(s,t) = W(s',t)$. If not, we could find an accumulation point $s' \in [-T,T]$ where $U(s,t) = W(s',t)$ and $\partial_s W(s',t) = 0$, a contradiction with $(**)$. Let $s_1, \dots, s_N \in [-T,T]$ be the points such that $U(s_0,t_0) = W(s_j,t_0)$, $j = 1, \dots, N$.
Now we claim that for any $r > 0$ there exists $\delta > 0$ such that

$$\forall\, (s,t) \in V_\delta(s_0,t_0)\; \exists\, (s',t) \in \bigcup_{j=1}^{N} V_r(s_j,t_0) \quad : \quad U(s,t) = W(s',t).$$

If this would fail, we could find $r > 0$ and a sequence $(s^\nu,t^\nu) \to (s_0,t_0)$ such that for all ν and for all $(s',t^\nu) \in \bigcup_{j=1}^{N} V_r(s_j,t_0)$ we have $U(s^\nu,t^\nu) \neq W(s',t^\nu)$. On the other hand by $(***)$, there exists $s'^\nu \in [-T,T]$ such that $U(s^\nu,t^\nu) = W(s'^\nu,t^\nu)$ and now in particular $|s'^\nu - s_j| \geq r$ for all $j = 1, \dots, N$. Up to a subsequence we have $s'^\nu \to s'$ and $t^\nu \to t_0$ with $U(s_0,t_0) = W(s',t_0)$ and $s' \neq s_j$, $j = 1, .., N$, a contradiction. Define

$$\Sigma_j := \left\{(s,t) \in \overline{V}_\delta(s_0,t_0) \;\middle|\; \exists (s',t) \in \overline{V}_r(s_j,t_0) \;:\; U(s,t) = W(s',t)\right\}.$$

Then Σ_j is closed and $\overline{V}_\delta(s_0,t_0) = \Sigma_1 \cup \dots \cup \Sigma_N$. It follows from Baire's Theorem that one of the Σ_j, say Σ_1, has non-empty interior. Let $(\bar{s},\bar{t}) \in int(\Sigma_1)$ and denote by $(\bar{s}',\bar{t})$ a preimage $W^{-1}(U(\bar{s},\bar{t}))$ in $V_r(s_1,t_0)$. Let $0 < r_1 < r$ be such that $V_{r_1}(\bar{s}',\bar{t}) \subset V_r(s_1,t_0)$ and $0 < \delta_1 < \delta$ be such that $V_{\delta_1}(\bar{s},\bar{t}) \subset \Sigma_1$ and such that for all $(s,t) \in V_{\delta_1}(\bar{s},\bar{t})$ there exists $(s',t) \in V_{r_1}(\bar{s}',\bar{t})$ with $U(s,t) = W(s',t)$. It follows from our construction that for all $0 < h' \leq r_1$ there exists $0 < h \leq \delta_1$ such that for all $(s,t) \in V_h(\bar{s},\bar{t})$, there exists $(s',t) \in V_{h'}(\bar{s}',\bar{t})$ such that $U(s,t) = W(s',t)$. We can therefore apply Theorem 34 with $(s_0,t_0) = (\bar{s},\bar{t})$, $U_0 = U$, $U_1 = W(\cdot + \bar{s}' - \bar{s})$ and $h_0 = r_1$, to obtain $U_0 = U_1$. This implies that W is an $(\bar{s}' - \bar{s})$-shift of U, $\bar{s}' - \bar{s} \neq 0$, which contradicts Lemma 36. $\square$

Remark. Write $v(\pm\infty,t) := v^\pm(t)$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. By applying Proposition 37 to the symplectic symmetries $\sigma^1, \sigma^2, \dots, \sigma^k = id$, we find for a solution (v,η) of (3) with $im(v) \not\subset V_{fix}(\sigma^l)$ for $l = 1, \dots, k-1$ that the set

$$\mathcal{S}(U) = \left\{ (s,t) \in \mathbb{R} \times S^1 \;\middle|\; \begin{array}{rl} & (\partial_s v(s,t), \partial_s \eta(s)) \neq (0,0) \\ & (v(s,t),\eta(s)) \notin \big(\sigma^l \circ v(\overline{\mathbb{R}},t), \eta(\overline{\mathbb{R}})\big), \quad l = 1, \dots, k \\ \text{except} & (v(s,t),\eta(s)) = \big(\sigma^k \circ v(s,t), \eta(s)\big) \end{array} \right\}$$

is open and dense in $\{(s,t) \in \mathbb{R} \times S^1 \,|\, \partial_s v(s,t) \neq 0\}$. This follows from Baire's Theorem as the above set is the finite intersection of the open and dense sets $\mathcal{R}(U)$ and $\mathcal{S}_\sigma(U), \mathcal{S}_{\sigma^2}(U), \dots, \mathcal{S}_{\sigma^{k-1}}(U)$.

2.4. Transversality

In this subsection, we show that $\widehat{\mathcal{M}}(c^-, c^+, m)$ is a manifold for generic choices of J. We do so by describing this space as the zero-set of a Fredholm section $\mathcal{F}$ in a suitable Banach bundle $\mathcal{E} \to \mathcal{B}$. Then, we generalize this result to situations with a symplectic symmetry σ of finite order and show again that $\widehat{\mathcal{M}}(c^-, c^+, m)$ is a manifold, now for generic choices of J in the space of σ-symmetric almost complex structures.

Suppose we have chosen a Morse function h and a metric g_h on $crit\big(\mathcal{A}^H\big)$ such that (h, g_h) is Morse-Smale. Let ϕ denote the gradient flow of h with respect to g_h. It defines an $\mathbb{R}$-family of diffeomorphisms $T_h(t) \in Diff\big(crit\big(\mathcal{A}^H\big)\big)$ by

$$T_h(t)(x) := \phi^t(x).$$

A trajectory with m cascades from c^- to c^+, $c^\pm \in \mathrm{crit}(h)$ is by Definition 19 a tupel

$$(x, t) = \big((x_k)_{1\le k\le m},\, (t_k)_{1\le k\le m-1}\big),$$

where $x_k = (v_k, \eta_k)$ are non-constant $\mathcal{A}^H$-gradient trajectories and $t_k \geq 0$ non-negative real numbers satisfying some asymptotic and connectedness conditions. Using T_h, we can rewrite these conditions as follows:

$$\begin{array}{ll}(\boldsymbol{Asymptotics}) & \forall\, k\, \exists\, x_k^\pm \in crit\big(\mathcal{A}^H\big)\,, s.t. \lim\limits_{s\to\pm\infty} x_k = x_k^\pm \\ (\boldsymbol{Connectedness}) & \lim\limits_{t\to-\infty} T_h(t)(x_1^-) = c^-,\ \lim\limits_{t\to\infty} T_h(t)(x_m^+) = c^+,\ T_h(t_k)(x_k^+) = x_{k+1}^-.\end{array}$$

Note that the map $t \mapsto T_h(t)(x_k^+), t \in [0, t_k]$, is exactly the h-gradient trajectory y_k of the original definition. We call a trajectory with m cascades ***stable***, if for all t_k holds $t_k > 0$. In particular, trajectories with 0 or 1 cascade are always stable. We denote the space of stable trajectories with m cascades from c^- to c^+ by $\widehat{\mathcal{M}}_s(c^-, c^+, m)$ and the corresponding moduli space of unparametrized trajectories by $\mathcal{M}_s(c^-, c^+, m)$.

The dimension of $\widehat{\mathcal{M}}_s(c^-, c^+, m)$ is given in terms of Morse- and Conley-Zehnder indices. In order to define the Conley-Zehnder index as simple as possible, we assume throughout this section that the following map between fundamental groups induced by inclusion is injective

$$i_* : \pi_1(\Sigma) \to \pi_1(V).$$

Moreover, we consider only contractible closed Reeb orbits. Alternatively, we could assume that Σ is simply connected, i.e. $\pi_1(\Sigma) = 0$. Under these assumptions, we can choose for any closed Reeb orbit $v \in \mathcal{P}(\alpha) = crit\big(\mathcal{A}^H\big)$ a map $\bar{v} \in C^\infty(D, \Sigma)$ from the unit disc $D = \big\{z \in \mathbb{C}\,\big|\,|z| \le 1\big\}$ to Σ such that $\bar{v}(e^{2\pi i t}) = v(t)$. We call such a $\bar{v}$ a ***capping*** for v. Given a capping, we define in Section 3.3 the (transversal) Conley-Zehnder index $\mu(v, \bar{v})$ of the pair $(v, \bar{v})$.
Now we are ready to state the following fundamental theorem.

Theorem 38 (Global Transversality Theorem).
Given a Morse-Smale pair (h, g_h) on $crit\big(\mathcal{A}^H\big)$, there exists a set of second category $\mathcal{J}_{reg}$ of admissible families of smooth almost complex structures $J_t(\cdot, n)$ such that for every $J \in \mathcal{J}_{reg}$ the space of stable $\mathcal{A}^H$-trajectories $\widehat{\mathcal{M}}_s(c^-, c^+, m)$ has the structure of a finite dimensional manifold for every $c^-, c^+ \in crit(h)$ and $m \in \mathbb{N}$. Its local dimension at a trajectory with cascades (v, t) is given by

$$\begin{aligned}
\dim_{(v,t)} \widehat{\mathcal{M}}_s(c^-, c^+, m) = \quad & \Big(\mu_{CZ}(c^+, \bar{c}^+) + ind_h(c^+) - \frac{1}{2}\dim_{c^+}(crit\big(\mathcal{A}^H\big))\Big) \\
& - \Big(\mu_{CZ}(c^-, \bar{c}^-) + ind_h(c^-) - \frac{1}{2}\dim_{c^-}(crit\big(\mathcal{A}^H\big))\Big) \\
& + m - 1 + \sum_{k=1}^{m} 2c_1(\bar{v}_k^- \# v_k \# \bar{v}_k^+) \\
= \quad & \mu(c^+, \bar{c}^+) - \mu(c^-, \bar{c}^-) + m - 1 + \sum_{k=1}^{m} 2c_1(\bar{v}_k^- \# v_k \# \bar{v}_k^+), \\
\textit{where} \qquad \mu(c, \bar{c}) := \quad & \mu_{CZ}(c, \bar{c}) + ind_h(c) - \frac{1}{2}\dim_c(crit\big(\mathcal{A}^H\big)) + \frac{1}{2}.
\end{aligned}$$

Here, $\bar{v}_k^\pm$ are cappings for $v_k^\pm = \lim_{s \to \pm\infty} v_k$ and $\bar{c}^\pm$ are cappings for $c^\pm$ such that $\bar{v}_k^+ = \bar{v}_{k+1}^- \# \mathscr{C}_{k,k+1}$ where $\mathscr{C}_{k,k+1} : S^1 \times [0, t_k] \to \mathcal{N}^{\eta_k}$ is the cylinder given by $t \mapsto T_h(t)(v_k^+)$. Moreover $\bar{c} = \bar{v}_1 \# \mathscr{C}_{c^-,1}$ and $\bar{v}_m^+ = \bar{c}^+ \# \mathscr{C}_{m,c^+}$, where $\mathscr{C}_{c^-,1}$ and $\mathscr{C}_{m,c^+}$ are the analogue cylinders at the both ends. Finally $\bar{v}_k^- \# v_k \# \bar{v}_k^+$ is the sphere obtained by capping the cylinder v_k with $v_k^\pm$, c_1 is the first Chern-class of TV and $\dim_{c^\pm}(crit\big(\mathcal{A}^H\big))$ is the local dimension of $crit\big(\mathcal{A}^H\big)$ near $c^\pm$.

Remark.

- The condition $\bar{v}_k^+ = \bar{v}_{k+1}^- \# \mathscr{C}_{k,k+1}$ guarantees $\mu_{CZ}(v_k^+, \bar{v}_k^+) = \mu_{CZ}(v_{k+1}^-, \bar{v}_{k+1}^-)$.
- For $m = 0$, the dimension formula is not quite right. In this case $m - 1$ has to be replaced by zero, so that the dimension is given by $ind_h(c^+) - ind_h(c^-)$ just as in ordinary Morse theory.

The proof of Theorem 38 will make use of Fredholm theory and the implicit function theorem on infinite-dimensional Banach manifolds (see [36], app. A). We start by describing the analytic setup that we need.
Fix connected components $C^\pm$ of $crit\big(\mathcal{A}^H\big)$, fix J and choose $\delta := \delta(C^-, C^+) > 0$ such that $\delta/p < \rho$, where ρ is the constant for (C^-, C^+) from Proposition 32. We define

$$\mathcal{B} := \mathcal{B}(C^-, C^+) := \mathcal{B}_\delta^{1,p}(C^-, C^+), \qquad (v, \eta) \in \mathcal{B},\ v : \mathbb{R} \times S^1 \to V,\ \eta : \mathbb{R} \to \mathbb{R},$$

to be the Banach manifold of maps v which are locally in $W^{1,p}$, converge at both ends in $C^\pm$ and are in the weighted Sobolev spaces $W_\delta^{1,p}$ (see Definition 31), i.e.

1. (v, η) converges uniformly as $s \to \pm\infty$ to $(v^\pm, \eta^\pm) \in C^\pm$

2. there exist tubular neighborhoods $U^\pm$ of $v^\pm$ together with smooth parametrizations $\psi^\pm : U^\pm \times \mathbb{R} \to S^1 \times \mathbb{R}^{2n-1} \times \mathbb{R}$ such that for $s_0 > 0$ sufficiently large holds

$$\begin{aligned}&\psi^\pm \circ (v^\pm(t), \eta^\pm) = (t, 0)\\&\psi^+ \circ (v, \eta) - \psi^+ \circ (v^+, \eta^+) \in W^{1,p}_\delta\big([s_0, \infty) \times S^1\big) \times W^{1,p}_\delta\big([s_0, \infty)\big)\\&\psi^- \circ (v, \eta) - \psi^- \circ (v^-, \eta^-) \in W^{1,p}_\delta\big((-\infty, -s_0] \times S^1\big) \times W^{1,p}_\delta\big((-\infty, -s_0]\big).\end{aligned}$$

Note that $\delta/p < \rho$ implies together with the asymptotic estimates of Section 2.2 that any $\mathcal{A}^H$-gradient trajectory (v, η) which converges at the ends to $C^\pm$ is a point in $\mathcal{B}$. Using the exponential map with respect to any metric on V, it is not difficult to see that $\mathcal{B}$ is a Banach manifold whose differentiable structure does not depend on this metric. Observe that there are two natural smooth evaluation maps

$$ev^\pm : \mathcal{B} \to C^\pm, \qquad\qquad ev^\pm(v, \eta) = (v^\pm, \eta^\pm).$$

Let $(v, \eta) \in \mathcal{B}$. The tangent space $T_{(v,\eta)}\mathcal{B}$ at (v, η) can be identified with tuples

$$(\mathfrak{v}, \hat{\eta}) = (\mathfrak{v}_0, \hat{\eta}, \mathfrak{v}^-, \mathfrak{v}^+) \in W^{1,p}_\delta(\mathbb{R} \times S^1, v^*TV) \oplus W^{1,p}_\delta(\mathbb{R}, \mathbb{R}) \oplus T_{(v^-,\eta^-)}C^- \oplus T_{(v^+,\eta^+)}C^+.$$

The first two summands are naturally given as $\ker d(ev^-) \cap \ker d(ev^+)$, while the identification of the complement with $T_{(v^-,\eta^-)}C^- \oplus T_{(v^+,\eta^+)}C^+$ depends on the choice of submanifold charts for $C^\pm$.

Let $\mathcal{E}$ be the Banach bundle over $\mathcal{B}$ whose fiber at $(v, \eta) \in \mathcal{B}$ is given by

$$\mathcal{E}_{(v,\eta)} := L^p_\delta(\mathbb{R} \times S^1, v^*TV) \times L^p_\delta(\mathbb{R}, \mathbb{R}).$$

Consider the J-depending section

$$\mathcal{F} : \mathcal{B} \to \mathcal{E}, \qquad (v, \eta) \mapsto \big(\partial_s(v, \eta) - \nabla \mathcal{A}^H(v, \eta)\big) = \begin{pmatrix} \partial_s v + J_t(v, \eta)\big(\partial_t v - \eta X_H(v)\big) \\ \partial_s \eta + \int_0^1 H(v)dt \end{pmatrix}.$$

Note that $\mathcal{F}(v, \eta) = 0$ means that (v, η) satisfies the Rabinowitz-Floer equation (3) for an $\mathcal{A}^H$-gradient trajectory. Thus, we find that $\mathcal{F}^{-1}(0)$ is exactly the space of all $\mathcal{A}^H$-gradient trajectories which converge at both ends in $C^\pm$. Let us denote by $\widehat{\mathcal{M}}(C^-, C^+)$ the space of all $\mathcal{A}^H$-gradient trajectories (v, η) with asymptotics $(v^\pm, \eta^\pm)$ on $C^\pm$. Then we have apparently that $\widehat{\mathcal{M}}(C^-, C^+) = \mathcal{F}^{-1}(0)$.

Theorem 39 (Local Transversality Theorem)**.**
There exists a set $\mathcal{J}_{reg}$ of second category of admissible families of almost complex structures $J_t(\cdot, n)$, such that for all $J \in \mathcal{J}_{reg}$ holds that the space $\widehat{M}(C^-, C^+)$ is a manifold for any connected components $C^\pm \subset crit\big(\mathcal{A}^H\big)$. Its local dimension at $(v, \eta) \in \widehat{M}(C^-, C^+)$ is given by

$$\dim_{(v,\eta)} \widehat{M}(C^-, C^+) = \mu_{CZ}(v^+, \bar{v}^+) - \mu_{CZ}(v^-, \bar{v}^-) + 2c_1(\bar{v}^- \# v \# \bar{v}^+) + \frac{\dim C^- + \dim C^+}{2},$$

where $\bar{v}^-, \bar{v}^+$ are cappings for $v^\pm = \lim\limits_{s \to \pm\infty} v$.

Proof: We will show for all $J \in \mathcal{J}_{reg}$ that $\mathcal{F}$ and the zero section in $\mathcal{E}$ intersect transversally. Then it follows (as in the finite dimensional case) from the implicit function theorem that $\mathcal{F}^{-1}(0)$ is a manifold. The proof is split in 3 parts

1. First, we show that the vertical differential $D_{(v,\eta)}$ of $\mathcal{F}$ at $(v,\eta) \in \mathcal{F}^{-1}(0)$ given by

$$
\begin{aligned}
D_{(v,\eta)} &:= D^V\mathcal{F}(v,\eta) : T_{(v,\eta)}\mathcal{B} \to \mathcal{E}_{(v,\eta)} \\
D_{(v,\eta)} &: \quad W^{1,p}_\delta(\mathbb{R}\times S^1, v^*TV) \times W^{1,p}_\delta(\mathbb{R},\mathbb{R}) \times T_{v^-}C^- \times T_{v^+}C^+ \\
&\quad \to L^{1,p}_\delta(\mathbb{R}\times S^1, v^*TV) \times L^{1,p}_\delta(\mathbb{R},\mathbb{R})
\end{aligned}
$$

is a Fredholm operator of index

$$
ind\big(D_{(v,\eta)}\big) = \mu_{CZ}(v^+,\bar{v}^+) - \mu_{CZ}(v^-,\bar{v}^-) + 2c_1(\bar{v}^-\#v\#\bar{v}^+) + \frac{\dim C^- + \dim C^+}{2}.
$$

Proof:
We consider the restriction $\bar{D}_{(v,\eta)}$ of $D_{(v,\eta)}$ to $W^{1,p}_\delta(\mathbb{R}\times S^1, v^*TV) \times W^{1,p}_\delta(\mathbb{R},\mathbb{R})$. Then $\bar{D}_{(v,\eta)}$ is the linearization (the first variation) of the differential operator in the Rabinowitz-Floer equation (3). Hence it may be written as

$$
\bar{D}_{(v,\eta)} = \partial_s + A(s),
$$

where $A^\pm := \lim\limits_{s\to\pm\infty} A(s) = \nabla^2\mathcal{A}^H(v^\pm,\eta^\pm)$ and $A(s)\left(\begin{smallmatrix}\mathfrak{v}\\ \hat{\eta}\end{smallmatrix}\right)$ is given in (8) as

$$
A(s)\left(\begin{smallmatrix}\mathfrak{v}\\ \hat{\eta}\end{smallmatrix}\right) = \begin{pmatrix} -\Big(\nabla_{\mathfrak{v}}J + \hat{\eta}(\partial_n J)\Big)(\dot{v} - \eta X_H) - J\Big(\nabla_{(\dot{v}-\eta X_H)}\mathfrak{v} - \eta[\mathfrak{v}, X_H] - \hat{\eta}X_H\Big) \\ -\int_0^1 dH(\mathfrak{v})dt \end{pmatrix}
$$

Let $\gamma_\delta(s) = e^{\delta\cdot\beta(s)\cdot s}$ be the weight functions from the definition of $W^{1,p}_\delta$. They define a continuous isomorphism

$$
\phi : W^{1,p}_\delta \to W^{1,p}, \quad f \mapsto \gamma_\delta \cdot f.
$$

Let $\tilde{D}_{(v,\eta)}$ be the conjugated operator between the unweighted Sobolev spaces, i.e.

$$
\tilde{D}_{(v,\eta)} : W^{1,p}(\mathbb{R}\times S^1, v^*TV) \times W^{1,p}(\mathbb{R},\mathbb{R}) \to L^p(\mathbb{R}\times S^1, v^*TV) \times L^p(\mathbb{R},\mathbb{R})
$$

with
$$
\tilde{D}_{(v,\eta)} = \phi\bar{D}_{(v,\eta)}\phi^{-1}.
$$

As $\tilde{D}_{(v,\eta)}$ and $\bar{D}_{(v,\eta)}$ are conjugated, they are simultaneously Fredholm and if so, they have the same index. For $\zeta = (\mathfrak{v}_0,\hat{\eta}) \in W^{1,p}(\mathbb{R}\times S^1, v^*TV) \times W^{1,p}(\mathbb{R},\mathbb{R})$ we calculate

$$
\begin{aligned}
\tilde{D}_{(v,\eta)}(\zeta) = \phi\bar{D}_{(v,\eta)}\phi^{-1}(\zeta) &= \phi\bar{D}_{(v,\eta)}(\gamma_{-\delta}\cdot\zeta) \\
&= \phi\big(\partial_s(\gamma_{-\delta}\cdot\zeta) + A(s)(\gamma_{-\delta}\cdot\zeta)\big) \\
&= \gamma_\delta\Big(\gamma_{-\delta}(\partial_s\zeta) + \gamma_{-\delta}\big(\delta\beta(s) + \delta\partial_s\beta(s)s\big)\zeta + \gamma_{-\delta}\cdot A(s)\zeta\Big) \\
&= \partial_s\zeta + \Big[A(s) + \delta\big(\beta(s) + \partial_s\beta(s)s\big)\cdot Id\Big]\zeta.
\end{aligned}
$$

Hence, $\tilde{D}_{(v,\eta)}$ is given by $\tilde{D}_{(v,\eta)} = \partial_s + B(s)$, where $B(s)$ is the operator $B(s) = A(s) + \delta\big(\beta(s) + \partial_s\beta(s)s\big)Id$. Then $B^\pm := \lim\limits_{s\to\pm\infty} B(s) = A^\pm \pm \delta\cdot Id$.

As $A^\pm = \nabla^2\mathcal{A}^H(v^\pm,\eta^\pm)$, they are symmetric and so are $B^\pm$. Moreover, for $|\delta|$ smaller than the absolute value of the smallest non-zero eigenvalue of $A^\pm$, it follows that the $B^\pm$ are invertible. If $p=2$, it is shown in [44] that $\tilde{D}$ is then a Fredholm operator. For general p, the ideas in [46] can be used to give a proof. The index formula for $D_{(v,\eta)}$ is proved in [14], Sec. 4.1. □

2. Let $\mathcal{J}^\ell := \mathcal{J}^\ell(\rho)$ be the open subset in the Banach manifold of admissible 1-periodic families of complex structures $J_t(\cdot,n)$ of class C^ℓ for which we may choose ρ as the constant from Proposition 32[2]. We consider the universal section

$$\mathcal{F}:\mathcal{J}^\ell\times\mathcal{B}\to\mathcal{E},\qquad \mathcal{F}(J,(v,\eta))=\begin{pmatrix}\partial_s v+J_t(v,\eta)\big(\partial_t v-\eta X_H(v)\big)\\ \partial_s\eta+\int_0^1 H(v)dt\end{pmatrix}$$

and we prove that the universal moduli space

$$\mathcal{U}^\ell := \mathcal{U}^\ell(C^-,C^+) := \big\{\big(J,(v,\eta)\big)\in\mathcal{J}^\ell\times\mathcal{B}\,\big|\,\mathcal{F}(J,(v,\eta))=0\big\}$$

is a separable Banach manifold of class C^ℓ, as $\mathcal{F}$ intersects the zero section transversally. For that, it suffices to show that the vertical differential

$$D_{(J,(v,\eta))}: T_J\mathcal{J}^\ell\times T_{(v,\eta)}\mathcal{B}\to\mathcal{E}_{(v,\eta)}$$

given by $$D_{(J,(v,\eta))}\Big(Y,(\mathfrak{v},\hat\eta)\Big)=D_{(v,\eta)}(\mathfrak{v},\hat\eta)+\begin{pmatrix}Y_t(v,\eta)\cdot(\partial_t v-\eta X_H)\\0\end{pmatrix}$$

is surjective for every $(J,(v,\eta))\in\mathcal{U}^\ell$. The tangent space $T_J\mathcal{J}^\ell$ consists of matrix valued maps $Y: S^1\times\mathbb{R}\to End(TV)$ of class C^ℓ satisfying the conditions

(1) $\omega(Yv,w)+\omega(v,Yw)=0 \qquad \forall\, v,w\in TV$

(2) $J_t(p,n)Y_t(p,n)+Y_t(p,n)J_t(p,n)=0 \qquad \forall\, p\in V, n\in\mathbb{R}, t\in S^1$

(3) $Y_t(p,n)=\begin{pmatrix}Y_\xi & 0\\ 0 & 0\end{pmatrix}$ on the cylindrical end of V, i.e. for $p\in[R,\infty)\times M$. Here, Y_ξ is independent of n and t and the block structure is with respect to the splitting $\xi\oplus span(R_\lambda,Y_\lambda)$

(4) $\sup_{n\in\mathbb{R}}||Y_t(\cdot,n)||_{C^\ell}<\infty$.

We have shown in 1.) that $D_{(v,\eta)}=D_{(J,(v,\eta))}|_{0\times T_{(v,\eta)}\mathcal{B}}$ is a Fredholm operator and has hence a finite dimensional cokernel. Therefore, $D_{J,(v,\eta)}$ has a finite dimensional cokernel as well and thus a closed range. Hence, for surjectivity it only remains to prove that the range is dense.

For that, consider q with $1/p+1/q=1$. The dual space $\big(\mathcal{E}_{(v,\eta)}\big)^*$ is then given by

$$\big(\mathcal{E}_{(v,\eta)}\big)^*=L^q_{-\delta}\big(\mathbb{R}\times S^1,v^*TV\big)\times L^q_{-\delta}\big(\mathbb{R},\mathbb{R}\big).$$

[2]Although ρ is arbitrary close to the absolute smallest non-zero eigenvalue of $\nabla^2\mathcal{A}^H$, it cannot be chosen globally for all J, as $\nabla^2\mathcal{A}^H$ depends on the metric g, which itself depends on J.

To show the density of the range, it suffices to show that any $(\zeta,\hat{\mu}) \in \big(\mathcal{E}_{(v,\eta)}\big)^*$ which annihilates the range is identically zero, i.e. we show that $(\zeta,\hat{\mu})=0$, if for all $J \in Y \in T_J\mathcal{J}^\ell$, $\mathfrak{v} \in W^{1,p}_\delta(\mathbb{R}\times S^1, v^*TV)$ and $\hat{\eta} \in W^{1,p}_\delta(\mathbb{R},\mathbb{R})$ holds that

$$\int_{\mathbb{R}\times S^1} \big\langle D_{J,(v,\eta)}(Y,\mathfrak{v},\hat{\eta}),(\zeta,\hat{\mu})\big\rangle\, ds\, dt = 0 \Leftrightarrow \begin{cases} \int \big\langle D_{(v,\eta)}(\mathfrak{v},\hat{\eta}),(\zeta,\hat{\mu})\big\rangle\, ds\, dt & = 0 \\ \int \langle Y(\partial_t v - \eta X_H),\zeta\rangle\, ds\, dt & = 0. \end{cases} \qquad (*)$$

The first equation implies that $(\zeta,\hat{\mu})$ is a weak solution of $D^*_{(v,\eta)}(\zeta,\hat{\mu})=0$, where $D^*_{(v,\eta)}$ is the formal adjoint of $D_{(v,\eta)}$ which is of the same form as $D_{(v,\eta)}$. As $D_{(v,\eta)}$ is elliptic, so is $D^*_{(v,\eta)}$ and it follows from elliptic regularity that hence $(\zeta,\hat{\mu})$ is of class C^ℓ and has the unique continuation property (in the sense of Theorem 33). As (v,η) is non-constant, we find that $\partial_s v \neq 0$. In Proposition 37, we showed that the set $\mathcal{R}(U)$ of regular points of $U=(v,\eta)$ is open and dense in the open non-empty set $\{(s,t)\in\mathbb{R}\times S^1 \,|\, \partial_s(v,\eta)\neq 0\}$. Recall that $\mathcal{R}(U)$ is given by

$$\mathcal{R}(U) := \left\{ (s,t)\in\mathbb{R}\times S^1 \;\middle|\; \begin{array}{c} (\partial_s v(s,t),\partial_s\eta(s)) \neq (0,0) \\ (v(s,t),\eta(s)) \neq (v^\pm(t),\eta^\pm) \\ (v(s,t),\eta(s)) \neq \big(v(s',t),\eta(s')\big), \forall s'\in\mathbb{R}\setminus\{s\} \end{array} \right\}.$$

Now, we consider the following open subset of $\mathcal{R}(U)$:

$$\Omega := \left\{(s,t)\in\mathcal{R}(U) \;\middle|\; v(s,t)\notin crit(H)\right\}.$$

As $\Sigma = H^{-1}(0)$ is a regular level set of H and $\lim_{s\to\pm\infty} v(s,\cdot)\in\Sigma$, it follows that Ω is also a non-empty open set. We will show that $(\zeta,\hat{\mu})$ vanishes identically on Ω and hence by the unique continuation property everywhere. Note that H is constant on $[R,\infty)\times M$ and hence $[R,\infty)\times M \subset critH$. Thus $v(\Omega)\subset V\setminus\big([R,\infty)\times M\big)$ and hence we have all freedom to perturb J on Ω, i.e. we do not have to pay attention to condition (3).

First, we prove that $\zeta\equiv 0$ on Ω. Suppose by contradiction that there exists $(s_0,t_0)\in\Omega$ such that $\zeta(s_0,t_0)\neq 0$. Set $p:=v(s_0,t_0)$ and $n:=\eta(s_0)$ and choose a linear map $Y_p : T_pV\to T_pV$ such that at the point (t_0,p,n) condition (1) and (2) for $Y\in T_J\mathcal{J}$ are satisfied and

$$\omega_p\big(Y_p J\partial_s v(s_0,t_0),\zeta(s_0,t_0)\big) > 0.$$

See for instance [36], Lem. 3.2.2, for an explicit construction of such a Y_p. Now choose an element $\widetilde{Y}\in T_J\mathcal{J}^\ell$ such that $\widetilde{Y}_{t_0}(p,n)=Y_p$. As (s_0,t_0) belongs to $\mathcal{R}(U)$, one can choose a smooth cutoff function $\beta : S^1\times V\times\mathbb{R}\to[0,1]$ supported near $\big(t_0,v(s_0,t_0),\eta(s_0)\big)=(t_0,p,n)$ such that for $Y:=\beta\cdot\widetilde{Y}\in T_J\mathcal{J}^\ell$ we have

$$\int_{\mathbb{R}\times S^1} \big\langle YJ\partial_s v,\zeta\big\rangle ds\, dt > 0.$$

This contradicts the second equation in $(*)$ and hence ζ has to vanish on Ω.

Secondly, we prove that also $\hat{\mu} \equiv 0$ on Ω. Suppose that $\mathfrak{v} \in W^{1,p}_\delta(\mathbb{R} \times S^1, v^*TV)$ is supported in Ω. By the first step, we may assume that $\zeta \equiv 0$ on Ω and this implies

$$\begin{aligned}\int_{\mathbb{R}\times S^1} \Big\langle D_{(v,\eta)}(\mathfrak{v}, 0), (\zeta, \hat{\mu})\Big\rangle ds\, dt = & \int_\Omega \Big\langle D_{(v,\eta)}(\mathfrak{v}, 0), (0, \hat{\mu})\Big\rangle ds\, dt \\ \overset{(8)}{=} & -\int_\Omega \hat{\mu}(s) \cdot dH\big(v(s,t)\big)\big[\mathfrak{v}(s,t)\big]\, ds\, dt.\end{aligned}$$

Now if there exists $(s_0, t_0) \in \Omega$ such that $\hat{\mu}(s_0, t_0) \neq 0$, then we can find a vector $\mathfrak{v}_p \in T_pV$ satisfying $dH_p(\mathfrak{v}_p) < 0$, as $v^{-1}(critH) \cap \Omega = \emptyset$. Using the above equation, we find that a suitable extension $\mathfrak{v}$ of $\mathfrak{v}_p$ gives

$$\int_{\mathbb{R}\times S^1} \Big\langle D_{(v,\eta)}(\mathfrak{v}, 0), (0, \hat{\mu})\Big\rangle ds\, dt > 0.$$

Thus, $\hat{\mu}$ also has to vanish on Ω. Now, it follows from the unique continuation property that $(\zeta, \hat{\mu}) = 0$ everywhere. Therefore, $D_{J,(v,\eta)}$ is surjective for all $(J, v, \eta) \in \mathcal{U}^\ell$. Thus, $\mathcal{F}$ is transversal to the zero section in $\mathcal{E}$ and it follows from the implicit function theorem on Banach spaces due to Smale (see for example [36], App. A) that $\mathcal{U}^\ell = \mathcal{F}^{-1}(0)$ is a separable C^ℓ-Banach manifold.

3. We prove the theorem. Consider the projection

$$\pi : \mathcal{U}^\ell \to \mathcal{J}^\ell, \qquad (J, v, \eta) \mapsto J.$$

Its differential at a point (J, v, η) is just the projection

$$d\pi(J, v, \eta) : T_{J,(v,\eta)}\mathcal{U}^\ell \to T_J\mathcal{J}^\ell, \qquad (Y, \xi, \hat{\eta}) \mapsto Y.$$

The kernel of $d\pi(J, v, \eta)$ is $0 \times (\ker D_{(v,\eta)})$ as $T_{J,(v,\eta)}\mathcal{U}^\ell = \ker D_{J,(v,\eta)}$ and $D_{J,(v,\eta)}$ restricted to $0 \times T\mathcal{B}$ is $D_{(v,\eta)}$. Moreover, it follows from linear algebra that $im(d\pi)$ has the same (finite) codimension as $im(D_{(v,\eta)})$. Indeed, if $Y \subset X$ is a subspace and $f : X \to Z$ a linear map, then there exists a natural ismorphism

$$f(X) \Big/ f(Y) \cong X \Big/ (\ker f + Y)\,.$$

Applied to our situation, we have on one hand

$$\begin{aligned}\operatorname{coker} d\pi = T\mathcal{J}^\ell \Big/ d\pi\big(T\mathcal{U}^\ell\big) &= d\pi\big(T\mathcal{J}^\ell \times T\mathcal{B}\big) \Big/ d\pi\big(\ker D_{J,(v,\eta)}\big) \\ &\cong T\mathcal{J}^\ell \times T\mathcal{B} \Big/ \big(0 \times T\mathcal{B} + \ker D_{J,(v,\eta)}\big)\,,\end{aligned}$$

and as $D_{J,(v,\eta)} : T\mathcal{J}^\ell \times T\mathcal{B} \to \mathcal{E}$ is surjective, we have on the other hand

$$\begin{aligned}\operatorname{coker} D_{(v,\eta)} = T\mathcal{E} \Big/ D_{(v,\eta)}\big(T\mathcal{B}\big) &= D_{J,(v,\eta)}\big(T\mathcal{J}^\ell \times T\mathcal{B}\big) \Big/ D_{J,(v,\eta)}\big(0 \times T\mathcal{B}\big) \\ &\cong T\mathcal{J}^\ell \times T\mathcal{B} \Big/ \big(0 \times T\mathcal{B} + \ker D_{J,(v,\eta)}\big)\,.\end{aligned}$$

Thus, $d\pi(J, v, \eta)$ is a Fredholm operator of the same index as $D_{(v,\eta)}$. In particular, π is a Fredholm map and it follows from the Sard-Smale theorem for ℓ sufficiently large, that the set $\mathcal{J}^{l,C^-,C^+}_{reg}$ of regular values of π is a set of second category.

Note that $J \in \mathcal{J}_{reg}^{l,C^-,C^+}$ is a regular value of π exactly if $D_{(v,\eta)}$ is surjective for every $(v,\eta) \in \mathcal{F}^{-1}(0)$. In other words, for every $J \in \mathcal{J}_{reg}^{l,C^-,C^+}$ is $\widehat{\mathcal{M}}(C^-,C^+) = \pi^{-1}(J)$ a manifold of dimension $ind(D_{(v,\eta)})$, again by the implicit function theorem.
It is possible to show that the set $\mathcal{J}_{reg}^{C^-,C^+}(\rho) := \mathcal{J}_{reg}^{l,C^-,C^+} \cap \mathcal{J}^\infty(\rho)$ of smooth regular J is of second category in $\mathcal{J}(\rho)$ with respect to the C^∞-topology via a standard trick due to Taubes (see [36], Thm. 3.1.6 or [25], Thm. A.13 for details). We obtain a set of second category within all admissible almost complex structures by the union

$$\mathcal{J}_{reg}^{C^-,C^+} := \bigcup_{\rho>0} \mathcal{J}_{reg}^{C^-,C^+}(\rho).$$

The countable intersection

$$\mathcal{J}_{reg} := \bigcap_{C^\pm \subset crit(\mathcal{A}^H)} \mathcal{J}_{reg}^{C^-,C^+}$$

is still a set of second category in the set of all admissible almost complex structures and has the property that for all $J \in \mathcal{J}_{reg}$ and all connected components $C^-, C^+ \subset crit\big(\mathcal{A}^H\big)$ holds that $\widehat{\mathcal{M}}(C^-,C^+)$ is a finite dimensional manifold. $\square$

Now we can give the proof of the Global Transversality Theorem 38.

Proof:

0. If there are 0 cascades, the theorem follows from ordinary Morse homology, as (h,g) is a Morse-Smale pair. Hence we may assume that $m \geq 1$.

1. Let $(x,t) \in \widehat{\mathcal{M}}_s(c^-,c^+,m)$ be an arbitrary stable trajectory with m cascades $x_k = (v_k,\eta_k)$ passing through connected components $C_k \subset crit\big(\mathcal{A}^H\big)$, $0 \leq k \leq m$, i.e.

$$\begin{aligned} \lim_{s\to-\infty} x_k &= x_k^- = (v_k^-,\eta_k^-) \in C_{k-1} \\ \lim_{s\to+\infty} x_k &= x_k^+ = (v_k^+,\eta_k^+) \in C_k \end{aligned} \qquad 1 \leq k \leq m.$$

In terms of Theorem 39, we may write this as $x_k = (v_k,\eta_k) \in \widehat{\mathcal{M}}(C_{k-1},C_k)$. Let $\mathcal{U}^\ell(C_{k-1},C_k)$ denote the universal moduli space of $\mathcal{A}^H$-gradient trajectories between C_{k-1} and C_k from the proof of Theorem 39, i.e.

$$\mathcal{U}^\ell(C_{k-1},C_k) := \Big\{(J,(v,\eta)) \in \mathcal{J}^\ell \times \mathcal{B}(C_{k-1},C_k) \,\Big|\, \mathcal{F}(J,(v,\eta)) = 0\Big\}$$

and consider the C^ℓ-Banach manifold

$$\widetilde{\mathcal{U}} := \mathcal{U}^\ell(C_0,C_1) \times ... \times \mathcal{U}^\ell(C_{m-1},C_m) \times \big(\mathbb{R}^+\big)^{m-1}.$$

We define the universal moduli space $\mathcal{U}_m^\ell := \mathcal{U}_m^\ell(C_0,\dots,C_m)$ for trajectories with m cascades passing through $C_0,\dots,C_m$ to be the sub-Banach manifold of $\widetilde{\mathcal{U}}$, where each factor has the same almost complex structure J, i.e. $\mathcal{U}_m^\ell$ consists of tuples

$$\Big((J,x_1),\dots,(J,x_m),t_1,\dots,t_{m-1}\Big), \qquad (J,x_k) \in \mathcal{U}(C_{k-1},C_k),\ t_k \in \mathbb{R}^+,$$

with J fixed for $1 \leq k \leq m$.

Recall that we have for each k the evaluation maps

$$ev^- : \widehat{\mathcal{M}}(C_{k-1}, C_k) \to C_{k-1}, \qquad x_k \mapsto x_k^-$$

$$ev^+ : \widehat{\mathcal{M}}(C_{k-1}, C_k) \to C_k, \qquad x_k \mapsto x_k^+.$$

Moreover, recall that we denote by $T_h(t_k)$ the time t_k gradient flow of h on $crit\big(\mathcal{A}^H\big)$ and that we have for $(x,t) \in \widehat{\mathcal{M}}_s(c^-, c^+, m)$ the asymptotic and connectedness conditions

$$T_h(t_k)(\underbrace{x_k^+}_{ev^+(x_k)}) = \underbrace{x_{k+1}^-}_{ev^-(x_{k+1})}, \; 1 \leq k \leq m-1, \qquad \text{and} \quad x_1^- = ev^-(x_1) \in W^u(c^-)$$

$$\text{and} \quad x_m^+ = ev^+(x_m) \in W^s(c^+).$$

Here, $W^u(c^-)$ and $W^s(c^+)$ are the unstable/stable manifolds of $c^\pm$ with respect to the Morse-Smale pair (h, g_h) on C_0 resp. C_m. Write for the moment $T_h^{t_k}$ instead of $T_h(t_k)$ and consider the following map

$$\begin{aligned} \psi : \qquad & \mathcal{U}_m^\ell(C_0, C_1, \dots, C_m) \to C_0 \times (C_1)^2 \times \dots \times (C_{m-1})^2 \times C_m \\ & \big(J, x_1, \dots, x_m, t_1, \dots, t_{m-1}\big) \mapsto \\ & \Big(ev^-(x_1), T_h^{t_1}\big(ev^+(x_1)\big), \dots, ev^-(x_{m-1}), T_h^{t_{m-1}}\big(ev^+(x_{m-1})\big), ev^-(x_m), ev^+(x_m)\Big). \end{aligned}$$

Consider in the target space the submanifold $A_m(c^-, c^+)$ consisting of tuples

$$(q_0, p_1, q_1, \dots, p_{m-1}, q_{m-1}, p_m) \quad \text{such that} \quad p_k = q_k \in C_k,$$

$$\text{and} \quad q_0 \in W^u(c^-), \; p_m \in W^s(c^+).$$

We show below that ψ is transverse to $A_m(c^-, c^+)$. This implies by the implicit function theorem that $\psi^{-1}\big(A_m(c^-, c^+)\big)$ is a submanifold of $\mathcal{U}_m^\ell$ of codimension

$$\begin{aligned} \operatorname{codim} \psi^{-1}\big(A_m(c^-, c^+)\big) &= \operatorname{codim} W^u(c^-) + \sum_{k=1}^{m-1} \dim C_k + \operatorname{codim} W^s(c^+) \\ &= \sum_{k=1}^{m-1} \dim C_k + ind_h(c^-) + \big(\dim C_m - ind_h(c^+)\big). \end{aligned}$$

Note that $\dim W^u(c^-) = \dim C^- - ind_h(c^-)$, as we use the positive gradient flow on $C^\pm$ (see Definition 18). Having this result, we consider again the projection

$$\pi : \psi^{-1}\big(A_m(c^-, c^+)\big) \to \mathcal{J}^\ell.$$

This is now a Fredholm map of exactly the required index, as

$$\begin{aligned} ind\, \pi = \quad & \sum_{k=1}^{m} ind\, D_{v_k} - \operatorname{codim} \psi^{-1}\big(A_m(c^-, c^+)\big) + \dim\left(\mathbb{R}^+\right)^{m-1} \\ = \quad & \mu_{CZ}(c^+, \bar{c}^+) - \mu_{CZ}(c^-, \bar{c}^-) + \sum_{k=1}^{m} 2c_1(\bar{v}_k^- \# v_k \# \bar{v}_k^+) + m - 1 \\ & + \frac{1}{2} \dim C_m + \frac{1}{2} \dim C_0 - \Big(\dim C_m - ind_h(c^+)\Big) - ind_h(c^-). \end{aligned}$$

Using the Sard-Smale theorem, we find for ℓ sufficiently large that the set $\mathcal{J}^\ell_{reg}(C_0, \dots, C_m)$ of regular values of π is a set of second category in $\mathcal{J}^\ell$. Using Taubes trick, taking the union over $\rho > 0$ and countable intersections over tuples $(C_0, \dots, C_m) \subset crit\big(\mathcal{A}^H\big)^m$ and $m \in \mathbb{N}$, we get a set of second category $\mathcal{J}_{reg}$ such that for all $J \in \mathcal{J}_{reg}$, all $m \in \mathbb{N}$ and all tuples $(C_0, \dots, C_m)$ holds that $\widehat{\mathcal{M}}_s(c^-, c^+, m) = \pi^{-1}(J) \subset \psi^{-1}\big(A_m(c^-, c^+)\big)$ is a finite dimensional manifold of dimension $ind\,\pi$ which proves the theorem.

2. To show that ψ is transverse to $A_m(c^-, c^+, J)$ it suffices to show for any k that $0 \times TC_{k-1} \times TC_k \times 0$ lies in the image of

$$d\psi = d(ev_1^-) \times \big(dT_h(t_k) \circ d(ev_1^+)\big) \times \dots \times d(ev_m^+).$$

As $dT_h(t_k)$ is an isomorphism and as $\mathcal{U}^\ell(C_{k-1}, C_k)$ embeds naturally into $\mathcal{U}^\ell_m$, it suffices to show that $ev^- \times ev^+ : \mathcal{U}^\ell(C^-, C^+) \to C^- \times C^+$ are submersions for each pair of connected components $C^\pm \subset crit\big(\mathcal{A}^H\big)$. Let $(J, (v, \eta)) \in \mathcal{U}^\ell(C^-, C^+)$ and let $\zeta \in T_{ev^-(v,\eta)}C^- \times T_{ev^+(v,\eta)}C^+$ be arbitrary. We have to show that there exists $(Y, \mathfrak{v}, \hat{\eta}) \in \ker D_{J,(v,\eta)}$ such that

$$d(ev^-) \times d(ev^+)\big(Y, \mathfrak{v}, \hat{\eta}\big) = \zeta.$$

Surely, $ev^- \times ev^+ : \mathcal{B}(C^-, C^+) \to C^- \times C^+$ is a submersion. Hence, we may choose some arbitrary $(\mathfrak{v}_0, \hat{\eta}_0) \in T_{(v,\eta)}\mathcal{B}(C^-, C^+)$ such that

$$d(ev^-) \times d(ev^+)\big(0, \mathfrak{v}_0, \hat{\eta}_0\big) = \zeta.$$

In the proof of Theorem 39, (2.), we showed that $D_{J,(v,\eta)}$ as an operator with domain

$$T_J\mathcal{J}^\ell \oplus W^{1,p}_\delta(\mathbb{R} \times S^1, v^*TV) \oplus W^{1,p}_\delta(\mathbb{R}, \mathbb{R}) = \ker d(ev^-) \cap \ker d(ev^+)$$

is surjective, i.e. we did not use the $T_{(v^-,\eta^-)}C^- \oplus T_{(v^+,\eta^+)}C^+$ part of $T_{(v,\eta)}\mathcal{B}(C^-, C^+)$. Hence there exists $(Y_1, \mathfrak{v}_1, \hat{\eta}_1) \in T_J\mathcal{J}^\ell \oplus T_{(v,\eta)}\mathcal{B}(C^-, C^+)$ such that

$$D_{J,(v,\eta)}(Y_1, \mathfrak{v}_1, \hat{\eta}_1) = D_{J,(v,\eta)}(0, \mathfrak{v}_0, \hat{\eta}_0) \qquad \text{and} \qquad d(ev^\pm)_{J,(v,\eta)}(Y_1, \xi_1, \hat{\eta}_1) = 0.$$

Now set $(Y, \mathfrak{v}, \hat{\eta}) := (0 - Y_1, \mathfrak{v}_0 - \mathfrak{v}_1, \hat{\eta}_0 - \hat{\eta}_1)$. Then, $(Y, \mathfrak{v}, \hat{\eta})$ lies in the kernel of $D_{J,(v,\eta)}$ and satisfies

$$d(ev^-) \times d(ev^+)_{J,(v,\eta)}\big(Y, \mathfrak{v}, \hat{\eta}\big) = \zeta. \qquad \square$$

Remark.

- There is a similar version of the Global Transversality Theorem if we consider homotopies, i.e. if H, J, h and g_h depend on s. In this case however, we can no longer assume that all $\mathcal{A}^H$-gradient trajectories are non-constant. That transversality along constant trajectories holds automatically is shown in Appendix C.

- To show that $\widehat{\mathcal{M}}(c^-, c^+, m)$ is a manifold, we still have to deal with the unstable trajectories, i.e. whose where $t_k = 0$ for at least one k. Here, one uses a gluing argument that gives $\widehat{\mathcal{M}}(c^-, c^+, m)$ the structure of a manifold with corners (Theorem 46). For details, see [25], appendix A, discussion after Cor. A.15.

Next, we want to extend the Transversality Theorems 38 and 39 to situations where we have a smooth symplectic symmetry σ on V such that $\sigma^k = Id$ for some finite k. Recall that σ is a symplectic symmetry if it is a diffeomorphism of V such that $\sigma^*\lambda = \lambda$. We assume that the Hamiltonian H, the Morse function h and the metric g on $crit\big(\mathcal{A}^H\big)$ are chosen σ-invariant, i.e. $H(\sigma(p)) = H(p)$ for all $p \in V$, $h(\sigma(p)) = h(p)$ for all $p \in crit\big(\mathcal{A}^H\big)$ and $\sigma^* g = g$. Moreover, we assume that the following set is a symplectic submanifold of V:

$$V_{fix} := \Big\{ p \in V \,\Big|\, \sigma^l(p) = p \text{ for some } l, 1 \leq l \leq k-1 \Big\} = \bigcup_{1 \leq l \leq k-1} V_{fix}(\sigma^l).$$

An admissible almost complex structure J is called σ-symmetric if $\sigma^* J = J$. We denote the space of smooth σ-symmetric admissible almost complex structures by $\mathcal{J}^{sym}$.

Theorem 40 (Local Transversality Theorem with symmetry)**.**
Assume that there exists a set of second category $\mathcal{J}_{reg}^{fix}$ of admissable almost complex structures on V_{fix} such that for every $J \in \mathcal{J}_{reg}^{fix}$ the moduli space $\widehat{\mathcal{M}}(C^-, C^+)|_{V_{fix}}$ of solutions of the Rabinowitz-Floer equation (3) on V_{fix} is empty. Then there exists a set of second category $\mathcal{J}_{reg}^{sym} \subset \mathcal{J}^{sym}$ such that for all $J \in \mathcal{J}_{reg}^{sym}$ and all $C^\pm \subset crit\big(\mathcal{A}^H\big)$ holds that $\widehat{\mathcal{M}}(C^-, C^+)$ is a manifold of the same dimension as in Theorem 39.

The original idea of the following proof is due to Peter Uebele, who showed a similar result for symplectic homology in [50].

Proof: We proceed as in the proof of Theorem 39. Part (1.) remains completely unchanged, as the proof that $D_{(v,\eta)}$ is a Fredholm operator of the required index does not depend on the chosen almost complex structure.
In Part (2.), we now want to prove that the following universal moduli space is a C^ℓ-Banach manifold:

$$\mathcal{U}_\sigma^\ell := \Big\{ \big(J, (v,\eta)\big) \in \mathcal{J}^{\ell,sym} \times \mathcal{B} \,\Big|\, \mathcal{F}\big(J, (v,\eta)\big) = 0, im(v) \not\subset V_{fix} \Big\}.$$

Note that $\mathcal{U}_\sigma^\ell$ is in general not $\mathcal{F}^{-1}(0)$, as a priori we cannot exclude $im(v) \subset V_{fix}$. However, we clearly have that $\mathcal{U}_\sigma^\ell$ is an open subset of $\mathcal{F}^{-1}(0)$. Hence it suffices again to prove that $D_{J,(v,\eta)}$ is onto for every $(J, (v,\eta)) \in \mathcal{U}_\sigma^\ell$. As $D_{(v,\eta)}$ is Fredholm, $D_{J,(v,\eta)}$ has a closed range and it still suffices to show that its range is dense.
Now, the tangent space to $\mathcal{J}^{\ell,sym}$ consists of matrix valued functions $Y : S^1 \times \mathbb{R} \to End(TV)$ such that $Y \in T_J\mathcal{J}^\ell$ and Y is symmetric, i.e. $\sigma^* Y = Y$. Moreover, recall from the end of Section 2.3 the definition of regular and symmetric regular points $\mathcal{S}(U)$ of a solution $U = (v,\eta)$ of (3) with $im(v) \not\subset V_{fix}$:

$$\mathcal{S}(U) = \left\{ (s,t) \in \mathbb{R} \times S^1 \;\middle|\; \begin{array}{r} (\partial_s v(s,t), \partial_s \eta(s)) \neq (0,0) \\ (v(s,t), \eta(s)) \notin \left(\sigma^l \circ v(\overline{\mathbb{R}}, t), \eta(\overline{\mathbb{R}})\right), \quad l = 1, \dots, k \\ \text{except } \; (v(s,t), \eta(s)) = (\sigma^k \circ v(s,t), \eta(s)) \end{array} \right\}.$$

This set is open and dense in the open and non-empty set $\{(s,t) \in \mathbb{R} \times S^1 \,|\, \partial_s v \neq 0\}$ by Proposition 37. Moreover, by Lemma 35, we know that the set $F(U)$ of points (s,t) with $(v,\eta)(s,t) \notin V_{fix}$ is also open and dense in $\mathbb{R} \times S^1$. Thus we find that the set

$$\Omega := \left\{ (s,t) \in S_\sigma(U) \cap F(U) \;\middle|\; v(s,t) \notin crit(H) \right\}$$

is non-empty and open in $\mathbb{R} \times S^1$. As before, we show that every $(\zeta, \hat{\mu}) \in \left(\mathcal{E}_{(v,\eta)}\right)^*$ with

$$\int_{\mathbb{R}\times S^1} \left\langle D_{(v,\eta)}(\xi, \hat{\eta}), (\zeta, \hat{\mu}) \right\rangle ds\, dt = 0 \;\text{ and }\; \int_{\mathbb{R}\times S^1} \left\langle Y(\partial_t v - \eta X_H), \zeta \right\rangle ds\, dt = 0$$

for all $(Y, \xi, \hat{\eta}) \in T_J \mathcal{J}^{l,sym} \times T_{(v,\eta)}\mathcal{B}$ has to be zero on Ω and therefore everywhere by unique continuation. Assume that there exists $(s_0, t_0) \in \Omega$ with $\zeta(s_0, t_0) \neq 0$. Set $p := v(s_0, t_0) \in V$ and $n := \eta(s_0) \in \mathbb{R}$ and choose as before a linear map $Y_p : T_pM \to T_pM$ such that $Y_p \in T_J(\mathcal{J}^\ell)_{t_0}(p,n)$ and $\left\langle Y_p J \partial_s v(s_0,t_0), \zeta(s_0,t_0) \right\rangle > 0$. Using a cutoff function $\beta : S^1 \times V \times \mathbb{R} \to \mathbb{R}$ supported near (t_0, p, n) construct $\widetilde{Y} \in T_J \mathcal{J}^\ell$ supported near (t_0, p, n) such that

$$\int_{\mathbb{R}\times S^1} \left\langle \widetilde{Y} J \partial_s v, \zeta \right\rangle ds\, dt > 0. \tag{$*$}$$

Note that $\widetilde{Y} \notin T_J \mathcal{J}^{l,sym}$ in general, as it is not symmetric. Thus, we set

$$Y := \widetilde{Y} + \sigma^* \widetilde{Y} + (\sigma^2)^* \widetilde{Y} + \dots + (\sigma^{k-1})^* \widetilde{Y}.$$

Note that Y is supported near the k points $(t_0, \sigma^l(p), n)$, $0 \leq l \leq k-1$. For each $1 \leq l \leq k-1$ we have two cases:

- $\sigma^l(p) \notin im(v)$. Then $(*)$ still holds with Y instead of $\widetilde{Y}$ provided that $supp\,\beta$ is small enough in the V-direction.
- $\sigma^l(p) \in im(v)$. Then $(\sigma^l \circ v)(s_0,t_0) = v(s_1,t_1)$ for some $(s_1,t_1) \in \mathbb{R} \times S^1$. As $(s_0,t_0) \in \mathcal{S}(U)$ this implies either $\eta(s_1) \neq \eta(s_0) = n$ or that $t_1 \neq t_0$. Hence $(*)$ still holds for Y instead of $\widetilde{Y}$ provided that $supp\,\beta$ is small enough in the $\mathbb{R}$- resp. S^1-direction.

This shows that $\zeta \equiv 0$ on Ω. The proof that also $\hat{\mu} \equiv 0$ stays unchanged as it involves no perturbations on J. This shows that $\mathcal{U}^\ell_\sigma$ is a C^ℓ-Banach manifold.
Part (3.) of the proof works again mostly unchanged: Considering as before the projection $\pi_\sigma : \mathcal{U}^\ell_\sigma \to \mathcal{J}^{\ell,sym}$ we show again that $d\pi_\sigma$ is Fredholm of the same index as $D_{(v,\eta)}$ and that for ℓ large enough the set of regular values $\mathcal{J}^{\ell,sym}_{reg} \subset \mathcal{J}^{\ell,sym}$ of π is of second category. Using Taubes trick, we then obtain that the set $\widetilde{\mathcal{J}}^{sym}_{reg} := \mathcal{J}^{\ell,sym}_{reg} \cap \mathcal{J}^{\infty,sym}$ is of second category in $\mathcal{J}^{sym}$.

Attention: Note that even though $\pi^{-1}(J)$ is a manifold for $J \in \widetilde{\mathcal{J}}^{sym}_{reg}$ of the required dimension it might not be true that $\pi^{-1}(J) = \widehat{\mathcal{M}}(C^-, C^+)$ as $\mathcal{U}^\ell_\sigma$ only contains (v,η) with $im(v) \not\subset V_{fix}$. However, we assumed for all $J \in \mathcal{J}^{fix}_{reg}$ that the space $\widehat{\mathcal{M}}(C^-, C^+)|_{V_{fix}}$ is empty. Let $\mathcal{J}^{fix}$ denote the space of all smooth admissible almost complex structures on V_{fix} and recall that $\mathcal{J}^{fix}_{reg} \subset \mathcal{J}^{fix}$ is of second category. Note that the restriction of any $J \in \mathcal{J}^{sym}$ to V_{fix} gives an element in $\mathcal{J}^{fix}$. Hence we may consider the map

$$\varphi : \mathcal{J}^{sym} \to \mathcal{J}^{fix}, \qquad J \mapsto J|_{V_{fix}}.$$

Note that φ is continuous and open, as we can construct locally around any $J \in \mathcal{J}^{sym}$ a continuous map $\psi : \mathcal{J}^{fix} \to \mathcal{J}^{sym}$ such that $\varphi \circ \psi = Id$ and $\psi(J|_{V_{fix}}) = J$. The Lemma 41 shows then that $\varphi^{-1}\big(\mathcal{J}^{fix}_{reg}\big) \subset \mathcal{J}^{sym}$ is also a set of second category in $\mathcal{J}^{sym}$. Thus

$$\mathcal{J}^{sym}_{reg} := \widetilde{\mathcal{J}}^{sym}_{reg} \cap \varphi^{-1}\big(\mathcal{J}^{fix}_{reg}\big)$$

is of second category and we conclude that for $J \in \mathcal{J}^{sym}_{reg}$ there is no solution (v,η) of (3) lying entirely in V_{fix} and hence that $\widehat{\mathcal{M}}(C^-, C^+) = \pi^{-1}(J)$. $\square$

Lemma 41. *Let $\varphi : X \to Y$ be an open, continuous map between Baire spaces X, Y and let $A = \bigcap_{k\in\mathbb{N}} U_k$ be a countable intersection of open and dense sets $U_k \subset Y$, i.e. a set of second category in Y. Then $\varphi^{-1}(A)$ is of second category in X.*

Proof: As $\varphi^{-1}\big(\bigcap U_k\big) = \bigcap \varphi^{-1}\big(U_k\big)$, it suffices to show that the preimage $\varphi^{-1}(U)$ of an open and dense set $U \subset Y$ is open and dense in X. That $\varphi^{-1}(U)$ is open follows immediately from the fact that φ is continuous.
In order to show that $\varphi^{-1}(U)$ is dense, it suffices to show that for every open $V \subset X$ holds that $V \cap \varphi^{-1}(U) \neq \emptyset$. As φ is an open map, $\varphi(V)$ is open in Y. By density of U, we have that $U \cap \varphi(V) \neq \emptyset$ and hence that $\varphi^{-1}(U) \cap V \neq \emptyset$. $\square$

In order to give an appropriate version of the Global Transversality Theorem in a symmetric setup, let us make the following definition. Let $C^\pm \subset crit\big(\mathcal{A}^H\big)$ be any two connected components and let $c^\pm \in C^\pm \cap crit(h)$. We denote by $\widehat{\mathcal{M}}(C^-, c^+)$ the space of all $\mathcal{A}^H$ gradient trajectories (v,η) such that

$$\lim_{s\to-\infty}(v,\eta) = (v^-,\eta^-) \in C^- \quad \text{ and } \quad \lim_{s\to+\infty}(v,\eta) = (v^+,\eta^+) \in W^s(c^+) \subset C^+.$$

Similarly, we define $\widehat{\mathcal{M}}(c^-, C^+)$ as the space of $\mathcal{A}^H$-gradient trajectories (v,η) such that

$$\lim_{s\to-\infty}(v,\eta) = (v^-,\eta^-) \in W^u(c^-) \subset C^- \quad \text{ and } \quad \lim_{s\to+\infty}(v,\eta) = (v^+,\eta^+) \in C^+.$$

Note that both spaces are for all $J \in \mathcal{J}^{reg}$ manifolds of local dimensions

$$\begin{aligned} \dim_{(v,\eta)} \widehat{\mathcal{M}}(c^-, C^+) &= ind\, D_v - \dim W^u(c^-) \\ &= \mu_{CZ}(v^+,\bar{v}^+) - \mu_{CZ}(v^-,\bar{v}^-) \quad + \frac{\dim C^+ + \dim C^-}{2} - ind_h(c^-) \end{aligned} \tag{17}$$

$$
\begin{aligned}
\dim_{(v,\eta)} \widehat{\mathcal{M}}(C^-, c^+) &= ind\, D_v - \dim W^s(c^+) \\
&= \mu_{CZ}(v^+, \bar{v}^+) - \mu_{CZ}(v^-, \bar{v}^-) \quad + \frac{\dim C^+ + \dim C^-}{2} \\
&\qquad - \big(\dim C^+ - ind_h(c^+)\big) \\
&= \mu_{CZ}(v^+, \bar{v}^+) - \mu_{CZ}(v^-, \bar{v}^-) \quad + \frac{\dim C^- - \dim C^+}{2} + ind_h(c^+).
\end{aligned} \tag{18}
$$

This follows as in the proof of Theorem 38. Indeed, consider the map

$$
\psi : \mathcal{U}^\ell(C^-, C^+) \to C^- \times C^+, \qquad \big(J, (v,\eta)\big) \mapsto \big(ev^-(v,\eta), ev^+(v,\eta)\big).
$$

Then we see that $\psi^{-1}\big(C^- \times W^s(c^+)\big)$ resp. $\psi^{-1}\big(W^u(c^-) \times C^+\big)$ are submanifolds of $\mathcal{U}^\ell(C^-, C^+)$. Restricting the projection $\pi : \mathcal{U}^\ell(C^-, C^+) \to \mathcal{J}^\ell$ to these submanifolds, we then get that $\widehat{\mathcal{M}}(C^-, c^+)$ resp. $\widehat{\mathcal{M}}(c^-, C^+)$ are preimages of regular values of π.

Theorem 42 (Global Transversality Theorem with symmetry)**.**
Let $c^\pm = (v^\pm, \eta^\pm) \in crit(h)$. Assume that there exists a set of second category $\mathcal{J}_{reg}^{fix}$ of admissable almost complex structures on V_{fix} such that for all $J \in J_{reg}^{fix}$ and all $\eta^- < \eta_0 < \eta_1 < \eta^+$ holds that all moduli spaces $\widehat{\mathcal{M}}(\mathcal{N}^{\eta_0}, \mathcal{N}^{\eta_1})|_{V_{fix}}$ are empty. If c^- or c^+ lie in V_{fix}, assume that for all $J \in J_{reg}^{fix}$ and all $\eta^- < \eta_0 < \eta^+$ holds that $\widehat{\mathcal{M}}(c^-, \mathcal{N}^{\eta_0})|_{V_{fix}}$ resp. $\widehat{\mathcal{M}}(\mathcal{N}^{\eta_0}, c^+)|_{V_{fix}}$ are empty. If c^- and c^+ are both in V_{fix} assume that $\widehat{\mathcal{M}}(c^-, c^+, 1)|_{V_{fix}}$ is empty for all $J \in J_{reg}^{fix}$. Then there exists a set of second category $\mathcal{J}_{reg}^{sym}$ of admissable almost complex structures on V such that for all $J \in \mathcal{J}_{reg}^{sym}$ and all $c^\pm \in crit(h)$, $m \in \mathbb{N}$ holds that $\widehat{\mathcal{M}}(c^-, c^+, m)$ is a manifold of the same dimension as in Theorem 38.

Proof: Using the same arguments as in the proof of Theorem 38, it is obvious that we can find a set of second category $\mathcal{J}_{reg}^{sym}$ such that for every $J \in \mathcal{J}_{reg}^{sym}$ the space of all trajectories with cascades from c^- to c^+, where all cascades are not contained in V_{fix}, is a manifold of the required dimension. Simply replace in the proof $\mathcal{U}^\ell(C_{k-1}, C_k)$ by $\mathcal{U}^\ell_\sigma(C_{k-1}, C_k)$, use Theorem 40 instead of 39 and proceed otherwise in the same way.
It remains to verify that these trajectories with cascades are the only ones, i.e. we have to make sure that no cascade that is part of a trajectory between c^- and c^+ is contained in V_{fix}. It clearly suffice to assume for every pair $\eta_0, \eta_1 \in [\eta^-, \eta^+]$ that the space $\widehat{\mathcal{M}}(\mathcal{N}^{\eta_0}, \mathcal{N}^{\eta_1})|_{V_{fix}}$ of all gradient trajectories between $\mathcal{N}^{\eta_0}$ and $\mathcal{N}^{\eta_1}$ lying entirely in V_{fix} is empty. At the two ends however, we can weaken this assumption:

- If c^- or c^+ is not in V_{fix}, then it follows from the σ-symmetry of g that $W^u(c^-) \cap V_{fix} = \emptyset$ resp. $W^s(c^+) \cap V_{fix} = \emptyset$. Indeed, a point $p \in W^{s/u}(c^\pm) \cap V_{fix}$ would imply that $T_h(t)(p) \in V_{fix}$ for all t and hence that $\lim\limits_{s \to \pm\infty} T_h(t)(p) = c^\pm \in V_{fix}$. We know therefore that the first/last cascade cannot be contained in V_{fix} as one of its ends is not in V_{fix}.

- If c^- or c^+ lie in V_{fix} as well as the first/last cascade x_1 resp. x_m then we get from the symmetry of g that $x_1 \in \widehat{\mathcal{M}}(c^-, \mathcal{N}^{\eta_0})|_{V_{fix}}$ resp. $x_m \in \widehat{\mathcal{M}}(\mathcal{N}^{\eta_0}, c^+)|_{V_{fix}}$ as

$$T_h(t)\big(ev^-(x_1)\big) \in V_{fix} \quad \text{resp.} \quad T_h(t)\big(ev^+(x_m)\big) \in V_{fix} \quad \text{for all } t.$$

  However, our assumptions guarantee that this cannot happen.

- If c^- and c^+ lie in V_{fix} and we consider only trajectories with one cascade, which lies in V_{fix}, then the same arguments show that we actually consider the space $\widehat{\mathcal{M}}(c^-, c^+, 1)|_{V_{fix}}$ which we also assume to be empty. $\square$

Corollary 43. *Let $\sigma : V \to V$ be a symplectic symmetry of order k such that the set $V_{fix} = V \setminus \{p \,|\, \sigma^l(p) \neq p, 0 < l < k\}$ is a symplectic submanifold. Let H, h and g_h be σ-symmetric and suppose that $c^\pm = (v^\pm, \eta^\pm) \in crit(h) \cap V_{fix}$. Moreover assume that the assumptions of Theorem 42 are satisfied. Then it holds for all $J \in \mathcal{J}_{reg}^{sym}$ that the cardinality of the zero-dimensional component of $\mathcal{M}(c^-, c^+)$ is divisible by k.*

Proof: By Theorem 42, we know for $J \in \mathcal{J}_{reg}^{sym}$ that $\mathcal{M}(c^-, c^+)$ is a manifold. Its zero-dimensional component is by Theorem 44 compact and hence a finite set. Since H, J, h and g_h are σ-symmetric and $c^\pm \in V_{fix}$, it follows for any flow line with cascades (x, t) from c^- to c^+ that $(\sigma \circ x, t)$ is also a flow line with cascades from c^- to c^+. Since no flow line lies in V_{fix}, all following flow lines with cascades are pairwise different:

$$(\sigma^0 \circ x, t),\ (\sigma^1 \circ x, t), \dots, (\sigma^{k-1} \circ x, t).$$

This implies that $\#\mathcal{M}(c^-, c^+)$ is divisible by k. $\square$

3. Compactness and additional properties

3.1. Compactness

This subsection is (with a few changes) taken from [14]. We repeat it here for completeness and to give a generalization of Corollary 3.8, [14], at the end of this subsection. The main object is to show that the moduli spaces of $\mathcal{A}^{H_s}$-gradient trajectories for Hamiltonians H or homotopies H_s are compact. As the latter case includes the first, all proofs will be given for homotopies.
We still assume that (V,λ) is the completion of a compact Liouville domain $\tilde{V}$ with contact boundary M, that $\Sigma \subset V$ is a contact hypersurface bounding a compact Liouville domain W and that H is a defining Hamiltonian for Σ. We denote by $M \times \mathbb{R}$ the symplectization of M lying in V.

A homotopy of defining Hamiltonians is for us a smooth family $(H_s) \subset C^\infty(V)$ with $-\infty < s_- \leq s \leq s_+ < +\infty$, where all H_s are defining Hamiltonians of exact contact hypersurfaces $\Sigma_s := H_s^{-1}(0)$. We fix once and for all a smooth monotone cutoff function β satisfying $\beta(s) = s_-$ for $s \leq s_-$ and $\beta(s) = s_+$ for $s \geq s_+$. Using β, we extend the homotopy to $\mathbb{R}$ as

$$s \mapsto H_{\beta(s)}, \quad s \in \mathbb{R},$$

which is everywhere smooth and constant outside $[s_-, s_+]$. We set

$$H_- := H_s, \quad s \leq s_- \qquad \text{and} \qquad H_+ := H_s, \quad s \geq s_+.$$

To such a homotopy, we associate the following non-negative quantities:

$$||H||_\infty := \max_{x\in V,\, s\in[s_-,s_+]} |H_s(x)|, \qquad ||\dot{H}||_1 = \max_{x\in V} \int_{s_-}^{s_+} \left|\tfrac{d}{ds} H_s(x)\right| ds.$$

We define analog quantities for $H_{\beta(s)}$, $s \in \mathbb{R}$, which we can estimate by

$$||H_\beta||_\infty = ||H||_\infty, \qquad ||\dot{H}_\beta||_1 \leq ||\beta'||_\infty \cdot ||\dot{H}||_1. \tag{19}$$

By abuse of language, we say that an $\mathcal{A}^{H_s}$-gradient trajectory for a homotopy H_s is a solution of the s-dependent Rabinowitz-Floer equation

$$\begin{aligned} \partial_s v + J_t(v,\eta)\big(\partial_t v - \eta X_{H_s}(v)\big) &= 0 \\ \partial_s \eta \; + \int_0^1 H_{\beta(s)}\big(v(s,t)\big) dt &= 0, \end{aligned} \tag{20}$$

where X_{H_s} is the Hamiltonian vector field of $H_{\beta(s)}$. Note that this equation is not the gradient flow equation of $\mathcal{A}^{H_s}$! However, we continue to write by abuse of notation

$$\nabla \mathcal{A}^{H_s}(v,\eta) := \begin{pmatrix} -J_t(v,\eta)\big(\partial_t v - \eta X_{H_s}(v)\big) \\ -\int_0^1 H_{\beta(s)}\big(v(s,t)\big) dt \end{pmatrix}.$$

The following theorem implies that the moduli space of such trajectories connecting critical points $(v^\pm, \eta^\pm) \in crit\big(\mathcal{A}^{H_\pm}\big)$ is compact for sufficiently slow homotopies H_s.

Theorem 44. *Let c and ε be given by the Proposition 50 below and suppose that $H_s, s \in [s_-, s_+]$, is a smooth family of defining Hamiltonians satisfying the inequality*

$$\left(c + \frac{||H||_\infty}{\varepsilon}\right) ||\beta'||_\infty ||\dot{H}||_1 =: d < 1. \tag{21}$$

Assume furthermore that $w^\nu = (v^\nu, \eta^\nu) \in C^\infty(\mathbb{R} \times S^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$ is a sequence of $\mathcal{A}^{H_s}$-gradient trajectories for which there exist $a, b \in \mathbb{R}$ such that

$$\lim_{s \to -\infty} \mathcal{A}^{H_s}\big(w^\nu(s)\big) \geq a \quad \text{and} \quad \lim_{s \to +\infty} \mathcal{A}^{H_s}\big(w^\nu(s)\big) \leq b \qquad \forall\, \nu \in \mathbb{N}. \tag{22}$$

Then there exists a subsequence (w^{ν_j}) of (w^ν) and a $\mathcal{A}^{H_s}$-gradient trajectory w such that w^{ν_j} converges in the $C^\infty_{loc}(\mathbb{R} \times S^1, V) \times C^\infty_{loc}(\mathbb{R}, \mathbb{R})$-topology to w.

Remark.

- If $H_s = H$ is a constant homotopy, then condition (21) is empty, as $||\dot{H}||_1 = 0$. So Theorem 44 is applicable to any Hamiltonian H.
- If the sequence w^ν has fixed asymptotics $\lim_{s\to\pm\infty} w^\nu(s) = (v^\pm, \eta^\pm) \in crit\big(\mathcal{A}^{H_\pm}\big)$, then (22) holds with $a = \mathcal{A}^{H_-}(v^-, \eta^-) = \eta^-$ and $b = \mathcal{A}^{H_+}(v^+, \eta^+) = \eta^+$.

Proof: The theorem follows from the standard Gromov-compactness result, as soon as we have the following uniform bounds:

- an L^∞-bound on the loops $v^\nu \in \mathscr{L}$ (so that v^ν stays in a compact region in V),
- an L^∞-bound on the Lagrange multiplier $\eta^\nu \in \mathbb{R}$ (so that η^ν stays in a bounded region in $\mathbb{R}$),
- an L^∞-bound on the derivatives of the loops v^ν (i.e. excluding bubbling).

The support of X_{H_s} lies inside $V \setminus \big(M \times [R, \infty)\big)$ for some large R, as H_s is constant outside a compact set, independent from s. So, the first component of any $\mathcal{A}^{H_s}$-gradient trajectory (v, η) which enters $M \times [R, \infty)$ satisfies due to (20) the holomorphic curve equation.
With our choice of almost complex structures, we conclude that v cannot touch any level set $M \times \{r\}$, $r > R$, from inside (see [35], Lem. 2.4). As its asymptotics lie outside of $M \times [R, \infty)$, it has to remain in the compact set $V \setminus M \times [R, \infty)$ for all time. Alternatively, this result follows also from the Maximum Principle (see Lemma 90). This gives the L^∞-bound on v.
The bound on η is shown in the remainder of this section (see Corollary 51). It is here, where condition (21) is needed. As the symplectic form ω is exact, there are no non-constant J-holomorphic spheres in V. This excludes bubbling and hence the derivatives of v can be controlled (see [36]). $\square$

Before giving the L^∞-bound on η, let us quickly state the most important consequences of Theorem 44.

Corollary 45 (Gromov-Floer compactness). *Let w^ν be a sequence of $\mathcal{A}^{H_s}$-gradient trajectories with $\lim_{s\to\pm\infty} \mathcal{A}^{H_s}\big(w^\nu(s)\big) = \eta^\pm$ fixed. Then there exists a subsequence w^{ν_j} and $\mathcal{A}^{H_s}$-gradient trajectories $(w_k)_{1\le k\le l}$ and sequences of real numbers $s_k^{\nu_j}$ such that*

$$w^{\nu_j}(\cdot + s_k^{\nu_j}) \to w_k \quad \textit{in the } C^\infty_{loc}(\mathbb{R}\times S^1, V)\times C^\infty_{loc}(\mathbb{R},\mathbb{R})\textit{-topology and}$$

- $\lim_{s\to-\infty} \mathcal{A}^{H_s}(w_1) = \eta^-$ *and* $\lim_{s\to+\infty} \mathcal{A}^{H_s}(w_l) = \eta^+$
- $\lim_{s\to+\infty} w_k(s,t) = \lim_{s\to-\infty} w_{k+1}(s,t) \qquad \forall\, 1\le k\le l-1.$

For the proof of Corollary 45, see [25], proof of Thm. A.11 (ff. 69), or [45], Prop. 4.2. Using a glueing argument and this corollary, one then can prove the following theorem.

Theorem 46. *Let H_s be a homotopy between defining Hamiltonians H_- and H_+, let $h^\pm$ be Morse functions on $crit\big(\mathcal{A}^{H_\pm}\big)$ and let $c^\pm \in crit(h^\pm)$. The moduli space*

$$\mathcal{M}(c^-,c^+) := \bigcup_{m\in\mathbb{N}} \mathcal{M}(c^-,c^+,m)$$

of all trajectories from c^- to c^+ carries the structure of a manifold without boundaries.

Remark. Note that we allow here that the times t_k that we stay on the critical manifold C_k may be zero.

Definition 47 ([25], A.8). *Let H_s be a homotopy between defining Hamiltonians H_- and H_+, let $h^\pm$ be Morse functions on $crit\big(\mathcal{A}^{H_\pm}\big)$ and let $c^\pm \in crit(h^\pm)$. A* ***broken trajectory with cascades*** *from c^- to c^+*

$$\boldsymbol{w} = (w_j)_{1\le j\le l^-+l^+=l},\ l^-,l^+,l\in\mathbb{N},$$

consists of trajectories with cascades w_j from c_{j-1} to c_j for $0\le j\le l$ such that $c_0 = c^-$ and $c_l = c^+$ and $c_j \in crit(h^-)$ for $0\le j\le l^-$ and $c_j\in crit(h^+)$ for $l^-+1\le j\le l$.

Remark. By Definition 47 every (unbroken) trajectory with cascades w from c^- to c^+ is also a broken trajectory with cascades via $\boldsymbol{w} = (w)$.

The following theorem is an easy consequence of Theorem 46 and the usual compactness result in ordinary Morse theory.

Theorem 48 (Compactness Theorem).
The space $\overline{\mathcal{M}}(c^-,c^+)$ of all broken trajectories with cascades from c^- to c^+ carries the structure of a manifold with corners. Its interior is exactly given by the unbroken trajectories with cascades.

Remark. It follows from Theorem 48 that for the boundary of the 1-dimensional component $\overline{\mathcal{M}}^1(c^-,c^+)$ of $\overline{\mathcal{M}}(c^-,c^+)$ holds that

$$\partial\overline{\mathcal{M}}^1(c^-,c^+) = \textstyle\bigcup_{c\in crit(h)} \mathcal{M}^0(c^-,c)\times\mathcal{M}^0(c,c^+).$$

This statement implies by a standard argument in Floer theory that $\partial^F\circ\partial^F = 0$.

The topology of the manifolds in Theorem 46 and 48 is given by the so called Floer-Gromov convergence that we define now.

Definition 49 ([25], A.9)**.** *Let H_s, $H_\pm$, $h^\pm$ and $c^\pm$ be as in Definition 47. Suppose that $w^\nu, \nu \in \mathbb{N}$, is a sequence of trajectories with cascades all from c^- to c^+. We say that w^ν* ***Floer-Gromov converges*** *to a broken trajectory with cascades from c^- to c^+*

$$\boldsymbol{w} = (w_j)_{1\le j\le l}, \qquad w_j = \big((x_k^\nu)_{1\le k\le m^\nu}, (t_k^\nu)_{1\le k\le m^\nu-1}\big)$$

if one of the following conditions holds:

1. *If H_s is constant and $\mathcal{A}^H(c^-) = \mathcal{A}^H(c^+)$, then all w^ν and w_j are trajectories with zero cascades, i.e. ordinary h-Morse flow lines. Here, we require that there exist real numbers s_j^ν such that $w^\nu(\cdot + s_j^\nu)$ converges in the C^∞_{loc}-topology to w_j.*

2. *If H_s is non-constant or $\mathcal{A}^H(c^-) < \mathcal{A}^H(c^+)$, then all w^ν have at least one cascade. Here, we require:*

   a) *If $w_j \in C^\infty(\mathbb{R}, crit(\mathcal{A}^H))$ is a trajectory with zero cascades, then there exists a sequence of h-Morse flow lines $y_j^\nu \in C^\infty(\mathbb{R}, crit(\mathcal{A}^H))$ converging in C^∞_{loc} to w_j, a sequence of real numbers s_j^ν and a sequence of integers $k^\nu \in [1, m^\nu]$ such that either $\lim\limits_{s\to-\infty} x_{k^\nu}^\nu(s) = y_j^\nu(s_j^\nu)$ or $\lim\limits_{s\to\infty} x_{k^\nu}^\nu(s) = y_j^\nu(s_j^\nu)$.*

   b) *If w_j is a trajectory with at least one cascade, we write*

   $$w_j = \big((x_{i,j})_{1\le i\le m_j}, (t_{i,j})_{1\le i\le m_j-1}\big),\ m_j \ge 1.$$

   We require that there exist surjective maps $\gamma^\nu : \big[1, \sum_{p=1}^l m_p\big] \to [1, m^\nu]$, which are monotone increasing, i.e. $\gamma^\nu(\lambda_1) \le \gamma^\nu(\lambda_2)$ for $\lambda_1 \le \lambda_2$, and real numbers s_λ^ν for every $\lambda \in \big[1, \sum_{p=1}^l m_p\big]$, such that

   $$x^\nu_{\gamma^\nu(\lambda)}(\cdot + s^\nu_\lambda) \to x_\lambda \quad \text{in} \quad C^\infty_{loc},$$

   where $x_\lambda = x_{i,j}$ is such that $\lambda = \sum_{p=1}^j m_p + i$. For $\lambda \in \big[1, \sum_{p=1}^l m_p - 1\big]$, set

   $$\tau_\lambda = \begin{cases} t_{i,j}, & \text{if } \lambda = \sum_{p=1}^j m_p + i,\ 0 < i < m_j + 1 \\ \infty, & \text{if } \lambda = \sum_{p=1}^j m_p \end{cases}$$

   and $$\tau_\lambda^\nu = \begin{cases} t^\nu_{\gamma^\nu(\lambda)}, & \text{if } \lambda = \max\big\{\lambda' \in \big[1, \sum_{p=1}^l m_p - 1\big] : \gamma^\nu(\lambda') = \gamma^\nu(\lambda)\big\} \\ 0 & \text{otherwise.} \end{cases}$$

   Now, we require that $\lim\limits_{\nu\to\infty} \tau_\lambda^\nu = \tau_\lambda$.

In the remainder of this subsection, we prove the L^∞-bound on η for a solution of the Rabinowitz-Floer equation (3) satisfying (21).

Proposition 50. *Assume that H_s, $s \in [s_-, s_+]$, is a smooth family of defining Hamiltonians for exact contact hypersurfaces $\Sigma_s \subset V$. Then there exist constants $\varepsilon > 0$ and $c < \infty$ depending only on the set $\{H_s \,|\, s_- \leq s \leq s_+\}$ and not on the particular parametrization, such that for all solutions (v, η) of (20) the following implication holds:*

$$||\nabla \mathcal{A}^{H_s}(v, \eta)|| \leq \varepsilon \qquad \Rightarrow \qquad |\eta| \leq c \cdot \big(|\mathcal{A}^{H_s}(v, \eta)| + 1\big)\,.$$

Proof: The proof is organized in three steps.

Step 1 *There exist $\delta > 0$ and a constant $c_\delta < \infty$ with the following property: For every $(v, \eta) \in \mathscr{L} \times \mathbb{R}$ with $v(t) \in U^s_\delta := H_s^{-1}\big((-\delta, \delta)\big)$ for all $t \in S^1$ holds*

$$|\eta| \leq 2 \cdot |\mathcal{A}^{H_s}(v, \eta)| + c_\delta \cdot ||\nabla \mathcal{A}^{H_s}(v, \eta)||.$$

We start by choosing $\delta > 0$ so small, such that for all $s_- \leq s \leq s_+$ and all $x \in U^s_\delta$ holds

$$\lambda(X_{H_s}(x)) \geq \tfrac{1}{2} + \delta. \qquad\qquad \text{(possible, as } \lambda\big(X_{H_s}(x)\big) = 1 \text{ for } x \in H_s^{-1}(0))$$

We set $c_\delta := \max\limits_{s_- \leq s \leq s_+} 2 \cdot ||\lambda|_{U^s_\delta}||_\infty$ and calculate for v with $im(v) \subset U^s_\delta$ that

$$\begin{aligned}
\big|\mathcal{A}^{H_s}(v, \eta)\big| &= \left|\int_0^1 \lambda(\dot v) - \eta H_s(v) dt\right| \\
&= \left|\int_0^1 \lambda(\eta X_{H_s}) dt + \int_0^1 \lambda(\dot v - \eta X_{H_s}) dt - \int_0^1 \eta H_s(v) dt\right| \\
&\geq \left|\eta \int_0^1 \lambda(X_{H_s}) dt\right| - \left|\int_0^1 \lambda(\dot v - \eta X_{H_s}) dt\right| - \left|\int_0^1 \eta H_s(v) dt\right| \\
&\geq |\eta| \cdot \left(\frac{1}{2} + \delta\right) - \frac{c_\delta}{2} ||\dot v - \eta X_{H_s}|| - |\eta| \cdot \delta \\
&\geq \frac{|\eta|}{2} - \frac{c_\delta}{2} \big|\big|\nabla \mathcal{A}^{H_s}(v, \eta)\big|\big|\,.
\end{aligned}$$

Step 2 *For each $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that*

$$\big|\big|\nabla \mathcal{A}^{H_s}(v, \eta)\big|\big| \leq \varepsilon \qquad \Rightarrow \qquad v(t) \in U^s_\delta \quad \forall\, t \in S^1.$$

First assume that $v \in \mathscr{L}$ has the property that there exist $t_0, t_1 \in S^1$ such that $|H_s(v(t_0))| \geq \delta$ and $|H_s(v(t_1))| \leq \frac{\delta}{2}$. We claim that

$$\big|\big|\nabla \mathcal{A}^{H_s}(v, \eta)\big|\big| \geq \frac{\delta}{2\kappa} \qquad \forall \eta \in \mathbb{R} \tag{23}$$

$$\text{where} \qquad \kappa := \max_{s_- \leq s \leq s_+} \max_{x \in \bar U^s_\delta, t \in S^1} ||\nabla_t H_s(x)||,$$

with ∇_t being the gradient of H_s with respect to $g_t = \omega(\cdot, J_t \cdot)$, i.e. $\nabla_t H_s = J_t X_{H_s}$.

We prove (23) with the following estimate:

$$\begin{aligned}
\big|\big|\nabla\mathcal{A}^{H_s}(v,\eta)\big|\big| \geq \sqrt{\int_0^1 ||\dot{v}-\eta X_{H_s}(v)||^2 dt} &\geq \int_0^1 ||\dot{v}-\eta X_{H_s}(v)||dt \\
&\geq \int_{t_0}^{t_1} ||\dot{v}-\eta X_{H_s}(v)||dt \\
&\geq \frac{1}{\kappa}\int_{t_0}^{t_1} ||\nabla_t H_s(v)||\cdot||\dot{v}-\eta X_{H_s}(v)||dt \\
&\geq \frac{1}{\kappa}\int_{t_0}^{t_1} |g(\nabla_t H_s(v),\dot{v}-\eta X_{H_s}(v)|\,dt \\
&= \frac{1}{\kappa}\int_{t_0}^{t_1} |g(\nabla_t H_s(v),\dot{v})|\,dt \\
&= \frac{1}{\kappa}\int_{t_0}^{t_1} |dH_s(\dot{v})|\,dt \\
&= \frac{1}{\kappa}\int_{t_0}^{t_1} |\partial_t H_s(v)|\,dt \\
&\geq \frac{1}{\kappa}\left|\int_{t_0}^{t_1} \partial_t H_s(v)dt\right| \\
&= \frac{1}{\kappa}\,|H_s(v(t_1)) - H_s(v(t_0))| \\
&\geq \frac{1}{\kappa}\,(|H_s(v(t_1))| - |H_s(v(t_0))|) \\
&\geq \frac{\delta}{2\kappa}.
\end{aligned}$$

Now assume that for v holds $v(t) \in V \setminus U^s_{\delta/2}$ for all $t \in S^1$. Then, we estimate

$$\big|\big|\nabla\mathcal{A}^{H_s}(v,\eta)\big|\big| \geq \left|\int_0^1 H_s(v)\,dt\right| \geq \frac{\delta}{2} \qquad \forall\,\eta\in\mathbb{R}. \tag{24}$$

From (23) and (24), step 2 follows with $\qquad \varepsilon := \dfrac{\delta}{2\cdot\max\{1,\kappa\}}.$

Step 3 *Proof of the proposition*

Choose δ as in step 1, $\varepsilon = \varepsilon(\delta)$ as in step 2 and $c = \max\{2, c_\delta\cdot\varepsilon\}$. Assume that $||\nabla\mathcal{A}^{H_s}(v,\eta)|| \leq \varepsilon$. Step 1 and 2 then imply that

$$|\eta| \leq 2|\mathcal{A}^{H_s}(v,\eta)| + c_\delta||\nabla\mathcal{A}^{H_s}(v,\eta)|| \leq c\left(|\mathcal{A}^{H_s}(v,\eta)|+1\right). \qquad \square$$

Remark. The constants c and ε are precisely the constants in Theorem 44. The stated independence from the parametrization is easily to be seen by the definitions of δ, c_δ and κ in the proof, which do involve only the fixed H_s and in particular no derivatives in s. This implies that for an arbitrary path $p:\mathbb{R}\to[s_-,s_+]$, we may choose for the homotopy $H_{p(s)}$ the same constants ε, c as for H_s.

Corollary 51. *Let $H_s, s \in [s_-, s_+]$, be a homotopy of defining Hamiltonians, such that*

$$\left(c + \frac{||H||_\infty}{\varepsilon}\right) ||\beta'||_\infty ||\dot{H}||_1 = d < 1. \tag{21}$$

Let $\omega = (v, \eta)$ be an $\mathcal{A}^{H_s}$-gradient trajectory such that for $a, b \in \mathbb{R}$ holds

$$\lim_{s \to -\infty} \mathcal{A}^{H_s}\big(w(s)\big) = a, \qquad \lim_{s \to +\infty} \mathcal{A}^{H_s}\big(w(s)\big) = b \qquad \forall\, n \in \mathbb{N}.$$

Let ε and c be as in Proposition 50 and set $M := \frac{|a|+|b|+b-a}{2}$. Then, η is uniformly bounded by

$$||\eta||_\infty \leq \frac{1}{1-d}\left(c \cdot (M+1) + \frac{||H||_\infty (b-a)}{\varepsilon^2}\right). \tag{25}$$

Proof:

Step 1 *We estimate the action $\mathcal{A}^{H_s}(w)$ and the energy $E(w)$ by*

$$E(w) \leq b - a + ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1, \tag{26}$$

$$\big|\mathcal{A}^{H_s}\big(w(s)\big)\big| \leq \quad M \quad + ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1. \tag{27}$$

At first, we calculate the s-derivative of the action $\mathcal{A}^{H_s}$ as

$$\frac{d}{ds}\mathcal{A}^{H_{\beta(s)}}(v,\eta)(s) = \big|\big|\nabla \mathcal{A}^{H_s}(v,\eta)(s)\big|\big|^2 - \eta(s) \cdot \beta'(s) \int_0^1 \left(\frac{d}{ds}H\right)_{\beta(s)}(v)\, dt. \tag{28}$$

Then we have

$$\begin{aligned}
E(w) &= \int_{-\infty}^{\infty} \big|\big|\nabla \mathcal{A}^{H_s}(v,\eta)(s)\big|\big|^2\, ds \\
&\overset{(28)}{=} \int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{A}^{H_{\beta(s)}}(v,\eta)(s) ds + \int_{-\infty}^{\infty} \eta(s) \cdot \beta'(s) \int_0^1 \left(\frac{d}{ds}H\right)_{\beta(s)}(v)\, dt\, ds \\
&\leq b - a + \int_0^1 \int_{s_-}^{s_+} \eta(s) \cdot \beta'(s) \cdot \left(\frac{d}{ds}H\right)_{\beta(s)}(v)\, ds\, dt \\
&\leq b - a + ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1,
\end{aligned}$$

where we used that $\frac{d}{ds}H_s \neq 0$ only for $s_- \leq s \leq s_+$. Next, we estimate the action by

$$\begin{aligned}
(a) \quad \big|\mathcal{A}^{H_s}(w(\sigma))\big| &= \left|\int_{-\infty}^{\sigma} \frac{d}{ds}\mathcal{A}^{H_s}(w(s))\, ds + \lim_{s \to -\infty} \mathcal{A}^{H_s}\big(w(s)\big)\right| \\
&\overset{(28)}{\leq} |a| + \int_{-\infty}^{\sigma} \big|\big|\nabla \mathcal{A}^{H_s}(v,\eta)(s)\big|\big|^2\, ds + \int_{-\infty}^{\sigma} ||\eta||_\infty ||\beta'||_\infty \int_0^1 \left|\frac{d}{ds}H\right| dt ds \\
(b) \quad \big|\mathcal{A}^{H_s}(w(\sigma))\big| &= \left|\int_{\sigma}^{\infty} \frac{d}{ds}\mathcal{A}^{H_s}(w(s)) - \lim_{s \to \infty} \mathcal{A}^{H_s}\big(w(s)\big)\right| \\
&\leq |b| + \int_{\sigma}^{\infty} \big|\big|\nabla \mathcal{A}^{H_s}(v,\eta)(s)\big|\big|^2\, ds + \int_{\sigma}^{\infty} ||\eta||_\infty ||\beta'||_\infty \int_0^1 \left|\frac{d}{ds}H\right| dt ds \\
\overset{(a,b)}{\Rightarrow} \quad \big|\mathcal{A}^{H_s}(w(\sigma))\big| &\leq \frac{1}{2}\Big(|a| + |b| + E(w) + ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1\Big) \\
\overset{(26)}{\Rightarrow} \quad \big|\mathcal{A}^{H_s}(w(\sigma))\big| &\leq M + ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1.
\end{aligned}$$

Step 2 *We prove that η is bounded.*
For $s \in \mathbb{R}$ and ε as in Proposition 50, let $\tau(s) \geq 0$ be defined by

$$\tau(s) := \inf\Big\{\tau \geq 0 \;\Big|\; \big|\big|\nabla \mathcal{A}^{H_s}(w(s+\tau))\big|\big| < \varepsilon\Big\}.$$

Then we have

$$\tau(\sigma) = \int_\sigma^{\sigma+\tau(\sigma)} 1\, ds \leq \int_\sigma^{\sigma+\tau(\sigma)} \frac{1}{\varepsilon^2} \big|\big|\nabla \mathcal{A}^{H_s}(w(s+\tau))\big|\big|^2\, ds \leq \frac{E(w)}{\varepsilon^2}. \tag{29}$$

Using Proposition 50, (26),(27) and (29), we can now calculate

$$\begin{aligned}
|\eta(\sigma)| &\leq |\eta(\sigma+\tau(\sigma))| + \int_\sigma^{\sigma+\tau(\sigma)} \left|\frac{d}{ds}\eta(s)\right| ds \\
&\leq c \cdot \big(\big|\mathcal{A}^{H_{\sigma+\tau(\sigma)}}(v,\eta)\big| + 1\big) + \tau(\sigma)\cdot ||H||_\infty \\
&\leq c\cdot (M+1) + c\cdot ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1 + \frac{E(w)\cdot ||H||_\infty}{\varepsilon^2} \\
&\leq c\cdot (M+1) + \frac{||H||_\infty (b-a)}{\varepsilon^2} + \left(c + \frac{||H||_\infty}{\varepsilon^2}\right) ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1.
\end{aligned}$$

As σ is arbitrary, we obtain

$$||\eta||_\infty \leq c\cdot (M+1) + \frac{||H||_\infty (b-a)}{\varepsilon^2} + \left(c + \frac{||H||_\infty}{\varepsilon^2}\right) ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1.$$

Using the assumption (21), we conclude that

$$||\eta||_\infty \leq \frac{1}{1-d}\left(c\cdot (M+1) + \frac{||H||_\infty (b-a)}{\varepsilon^2}\right). \qquad \square$$

Corollary 52. *Fix $k > 1$ and assume that the constant d from (21) is so small that $d(k+1) < 1$. Let $(v^\pm, \eta^\pm)$ be critical points of $\mathcal{A}^{H_\pm}$ and suppose that there exists an $\mathcal{A}^{H_s}$-gradient trajectory (v,η) with $\lim\limits_{s\to\pm\infty}(v,\eta) = (v^\pm,\eta^\pm)$. Then:*

(a)

If for (v^-,η^-) holds $$\mathcal{A}^{H_-}(v^-,\eta^-) \geq \frac{dk}{1-d(k+1)} > 0, \tag{30}$$

then it holds for (v^+,η^+) that $$\mathcal{A}^{H_+}(v^+,\eta^+) \geq \left(1 - \frac{1}{k}\right)\cdot \mathcal{A}^{H_-}(v^-,\eta^-) > 0.$$

(b)

If for (v^+,η^+) holds $$\mathcal{A}^{H_+}(v^+,\eta^+) \leq \frac{-dk}{1-d(k+1)} < 0, \tag{31}$$

then it holds for (v^-,η^-) that $$\mathcal{A}^{H_-}(v^-,\eta^-) \leq \left(1 - \frac{1}{k}\right)\cdot \mathcal{A}^{H_+}(v^+,\eta^+) < 0.$$

Proof: We only prove (a), case (b) being completely analog. We first assume that the absolute value of the action at the positive asymptotic satisfies the inequality

$$\left|\mathcal{A}^{H_+}(v^+,\eta^+)\right| \leq \mathcal{A}^{H_-}(v^-,\eta^-). \tag{32}$$

Using the notation from the proof of Corollary 51, we write $a := \mathcal{A}^{H_-}(v^-,\eta^-)$ and $b := \mathcal{A}^{H_+}(v^+,\eta^+)$ and deduce

$$\begin{aligned} & b-a = \mathcal{A}^{H_+}(v^+,\eta^+) - \mathcal{A}^{H_-}(v^-,\eta^-) && \leq 0 \\ \text{and} \quad & M = \frac{|a|+|b|+b-a}{2} && \leq |b| \leq \mathcal{A}^{H_-}(v^-,\eta^-). \end{aligned}$$

Hence, we get from (25) and (30) the inequality

$$\begin{aligned} ||\eta||_\infty \leq \frac{c}{1-d}(M+1) &\leq \frac{c}{1-d}\left(\mathcal{A}^{H_-}(v^-,\eta^-)+1\right) \\ &\leq \frac{c}{1-d}\left(1+\frac{1-d(k+1)}{dk}\right)\mathcal{A}^{H_-}(v^-,\eta^-) = \frac{c}{kd}\cdot\mathcal{A}^{H_-}(v^-,\eta^-). \end{aligned}$$

This implies together with (21) and (26):

$$\begin{aligned} \mathcal{A}^{H_+}(v^+,\eta^+) &\overset{(26)}{\geq} \mathcal{A}^{H_-}(v^-,\eta^-) - ||\eta||_\infty ||\beta'||_\infty ||\dot{H}||_1 \\ &\geq \left(1-\frac{c}{kd}\cdot||\beta'||_\infty||\dot{H}||_1\right)\mathcal{A}^{H_-}(v^-,\eta^-) \overset{(21)}{\geq} \left(1-\frac{1}{k}\right)\mathcal{A}^{H_-}(v^-,\eta^-). \end{aligned}$$

This is the statement of the corollary under the additional assumption (32). Now assume that (32) does not hold. To prove the corollary, it suffices to exclude the case

$$b = \mathcal{A}^{H_+}(v^+,\eta^+) < -\mathcal{A}^{H_-}(v^-,\eta^-) = -a < 0. \tag{33}$$

We assume by contradiction that (33) holds. Then we obtain:

$$b-a<0 \qquad \text{and} \qquad M=0.$$

In particular, we get from (25) that $||\eta||_\infty \leq \dfrac{c}{1-d}$. Hence, we can estimate

$$\begin{aligned} \mathcal{A}^{H_-}(v^-,\eta^-) &\overset{(26)}{\leq} \mathcal{A}^{H_+}(v^+,\eta^+) + ||\eta||_\infty||\beta'||_\infty||\dot{H}||_1 \\ &\overset{(33)}{\leq} -\mathcal{A}^{H_-}(v^-,\eta^-) + \frac{c}{1-d}\cdot||\beta'||_\infty||\dot{H}||_1 \\ &\overset{(21)}{\leq} -\mathcal{A}^{H_-}(v^-,\eta^-) + \frac{d}{1-d} \\ &\overset{k>1}{\leq} -\mathcal{A}^{H_-}(v^-,\eta^-) + \frac{dk}{1-d(k+1)} \\ &\overset{(30)}{\leq} 0. \end{aligned}$$

But this implies that $\mathcal{A}^{H_-}(v^-,\eta^-) \leq 0$, which contradicts assumption (30). Hence, (33) has to be wrong and Corollary 52 follows. $\square$

3.2. Invariance and action filtration

The following theorems in this section will show that the Rabinowitz-Floer homology does not depend on any auxiliary structures. To be more precise: It will turn out, that $RFH(H,h)$ only depends on the exact contact filling W of (Σ,ξ). As mentioned in the introduction, we can hence write $RFH(W,\Sigma)$ instead.
Additionally, we define the Rabinowitz-Floer homology $RFH^{(a,b)}(W,\Sigma)$ for an action window (a,b). While $RFH^{(a,b)}(W,\Sigma)$ does depend on the contact form on Σ, we will show that $RFH^{(0^\pm,\infty)}(W,\Sigma)$ and $RFH^{(-\infty,0^\pm)}(W,\Sigma)$ are in fact invariant under Liouville isomorphisms and do therefore only depend on the exact contact filling W. The same holds true for the growth-rates $\Gamma^\pm(W,\Sigma)$, defined later in this section.

We start with the most basic invariance theorem:

Theorem 53 (**Frauenfelder & Cieliebak,[14]**).
$RFH(H,h)$ is independent of the almost complex structure J, the Morse function h and the metric g_h. Moreover, if $H_s, s\in[s_-,s_+]$, is a homotopy of defining Hamiltonians of contact hypersurfaces $\Sigma_s\subset V$, then $RFH(H_-,h_-)$ and $RFH(H_+,h_+)$ are canonically isomorphic.

Proof: The proof uses the usual arguments in Floer theory (see [25], Thm. A17). We show only the invariance of $RFH(H_s)$ under homotopies of defining Hamiltonians, as it involves some non-standard technical difficulties due to compactness. However, our proof should enable the reader to prove the invariance for J, h and g_h in the same manner by considering generic homotopies J_s, h_s and g_{h_s}.

Step 0
Without loss of generality, we assume that $s_-=0$ and $s_+=1$. If not, replace H_s by $\tilde{H}_s := H_{s_-+(s_+-s_-)s}$.

Step 1
Let $\varepsilon>0$ and $c>0$ be the constants for the homotopy H_s given by Proposition 50. At first assume that H_s satisfies the inequality

$$\left(c+\frac{||H||_\infty}{\varepsilon}\right)\cdot||\beta'||_\infty||\dot{H}||_\infty\leq\frac{1}{8}. \tag{34}$$

Here, H_s is extended to $\mathbb{R}$ as $H_s := H_{\beta(s)}$ as discussed before. The norm $||\dot{H}||_\infty$ is defined as $||\dot{H}||_\infty := \max|\frac{d}{ds}H_s(x)|$. Note that $||\dot{H}||_1\leq||\dot{H}||_\infty\cdot(s_+-s_-)=||\dot{H}||_\infty$ here, so that (34) implies (21) with $d\leq\frac{1}{8}$. Write again $\lim\limits_{s\to-\infty}H_s=H_0=:H_-$ and $\lim\limits_{s\to+\infty}H_s=H_1=:H_+$ and pick Morse-functions $h_\pm$ on the critical manifolds $crit\big(\mathcal{A}^{H_\pm}\big)$.
For two critical points $c^\pm\in crit(h_\pm)$ we consider the moduli space $\widehat{\mathcal{M}}(c^-,c^+)$ of trajectories with cascades, where exactly one cascade consists of an $\mathcal{A}^{H_s}$-gradient trajectory, while all other cascades are either $\mathcal{A}^{H_-}$- or $\mathcal{A}^{H_+}$-gradient trajectories. It follows from a parametric version of the Global Transversality Theorem 38 that $\widehat{\mathcal{M}}(c^-,c^+)$ is a manifold and it follows from the Compactness Theorem 44 that its zero-dimensional component $\widehat{\mathcal{M}}^0(c^-,c^+)$ is a finite set.

We define a linear map $\phi : RFC(H_+, h_+) \to RFC(H_-, h_-)$ as the linear extension of

$$\phi(c^+) := \sum_{c^- \in crit(h_-)} \#_2 \widehat{\mathcal{M}}^0(c^-, c^+) \cdot c^-, \qquad c^+ \in crit(h_+). \tag{35}$$

<u>*Claim*</u>: The sum on the right hand side satisfies the finiteness condition (4).
<u>*Proof*</u>: Condition (34) on H_s, i.e. $d = \frac{1}{8}$, guarantees that Corollary 52 can be applied with $k = 2$. Hence $\widehat{\mathcal{M}}^0(c^-, c^+) \neq \emptyset$ implies

- $\mathcal{A}^{H_-}(c^-) \leq \max\left\{2\mathcal{A}^{H_+}(c^+), \frac{2d}{1-3d}\right\}$ if $\mathcal{A}^{H_+}(c^+) > -\frac{2d}{1-3d}$
- $\mathcal{A}^{H_-}(c^-) \leq \frac{1}{2}\mathcal{A}^{H_+}(c^+)$ otherwise.

The action of all c^- on the right hand side is therefore bounded from above in terms of $\mathcal{A}^{H_+}(c^+)$. Condition (4) follows therefore from the fact that the action spectrum of $\mathcal{A}^{H_-}$ is closed and discrete (Theorem 23). $\square$

Using again the Compactness Theorem 44, it follows by standard arguments in Floer theory (i.e. glueing, see [46] and [25]) that

$$\partial^- \circ \phi = \phi \circ \partial^+$$

so that ϕ induces a well-defined homomorphism on Floer homologies

$$\Phi : RFH(H_+, h_+) \to RFH(H_-, h_-).$$

The inverse homotopy $\bar{H}_s := H_{1-s}$ yields a homomorphism

$$\Psi : RFH(H_-, h_-) \to RFH(H_+, h_+).$$

For $R \geq 1$, we define the concatenation of H_s and $\bar{H}_s$ by the formula

$$K_s := H_s \#_R \bar{H}_s = \begin{cases} H_{s+R} & s \leq 0 \\ \bar{H}_{s-R} & s \geq 0 \end{cases}$$

which yields a homotopy K_s from H_- via H_+ back to H_-. Note that K_s is non-constant only for $-R \leq s \leq 1 - R$ and $R - 1 \leq s \leq R$. From (34) and Proposition 50 follows

$$\left(c + \frac{||K||_\infty}{\varepsilon^2}\right) \cdot ||\beta'||_\infty ||\dot{K}||_1 < \frac{1}{4}.$$

Note that we can use the same constants ε and c as for H_s and $\bar{H}_s$ as they depend by Proposition 50 only on the set $\{H_s\} = \{K_s\}$. Using again the Corollaries 51 and 52 and the standard gluing argument, we see that the composition

$$\Phi \circ \Psi : RFH(H_-, h_-) \to RFH(H_-, h_-)$$

is given by counting gradient flow lines of $\mathcal{A}^{K_s}$ (see again [46]).

Now, for $r \in [0,1]$ consider the homotopies of homotopies

$$H^r_s := H_{r\cdot s} \qquad \text{and} \qquad \bar{H}^r_s := \bar{H}_{r\cdot s} \qquad \text{and} \qquad K^r_s = H^r_s \#_r \bar{H}^r_s.$$

Then for each $r \in [0,1]$, the following estimate still continues to hold:

$$\left(c + \frac{||K^r||_\infty}{\varepsilon^2}\right) \cdot ||\beta'||_\infty ||\dot{K}^r||_1 < \frac{1}{4}.$$

Moreover, we have that $K^0_s = H_-$ does not depend on s any more and therefore induces the identity on $RFH(H_-, h_-)$. It follows that

$$\begin{aligned} & & \Phi \circ \Psi &= \text{identity on } \; RFH(H_-, h_-) \\ &\text{and similarly} & \Psi \circ \Phi &= \text{identity on } \; RFH(H_+, h_+). \end{aligned}$$

Thus, Φ is an isomorphism between $RFH(H_+, h_+)$ and $RFH(H_-, h_-)$. This finishes the proof under the additional assumption (34).

Step 2

Now consider a general homotopy H_s, so that (34) is not necessarily satisfied. Then we define for any $N \in \mathbb{N}$ and $0 \le j \le N-1$ the slower homotopies

$$H^{N,j}_s := H_{(j+s)/N} \quad \text{for } \; 0 \le s \le 1 \qquad \text{and} \qquad H^{N,j}_s := H^{N,j}_{\beta(s)} \quad \text{for } \; s \in \mathbb{R}. \tag{36}$$

We see that $||H^{N,j}||_\infty \le ||H||_\infty$ and $||\dot{H}^{N,j}||_\infty \le \frac{1}{N}||\dot{H}||_\infty$. Hence, we may choose $N \in \mathbb{N}$ so large, such that for each $0 \le j \le N-1$ holds

$$\left(c + \frac{||H^{N,j}||_\infty}{\varepsilon}\right) \cdot ||\beta'||_\infty ||\dot{H}^{N,j}||_\infty \le \frac{1}{8}.$$

Write $H_j := H^{N,j}_0 = H^{N,j-1}_1$ for the ends of the slow homotopies and choose Morse functions h_j for $crit\big(\mathcal{A}^{H_j}\big)$. Then, step 1 yields for $0 \le j \le N-1$ isomorphisms

$$\Phi_j : RFH(H_j, h_j) \to RFH(H_{j+1}, h_{j+1}).$$

The composition of these isomorphisms then shows that $RFH(H_0, h_0) = RFH(H_-, h_-)$ and $RFH(H_N, h_N) = RFH(H_+, h_+)$ are isomorphic. $\square$

The action $\mathcal{A}^H$ induces an $\mathbb{R}$-filtration on the chain complex $RFC(H,h)$. This allows us to define for $-\infty \le a \le b \le \infty$, $a, b \notin spec(\Sigma, \alpha)$ the following truncated chain complexes[3]:

$$RFC^{<b}(H,h) := \left\{ \sum_{crit(h)} \xi_c \cdot c \;\middle|\; \xi_c = 0 \text{ if } \mathcal{A}^H(c) \ge b \right\},$$

$$RFC^{(a,b)}(H,h) := RFC^{<b}(H,h) \Big/ RFC^{<a}(H,h) \cdot$$

Note that $RFC^{<\infty}(H,h)$ is the original chain complex $RFC(H,h)$.

[3]For variations on this definition, including $a, b \in spec(\Sigma, \alpha)$ see Section 6.3.

For $a \leq b \leq c$, we have the following natural short exact sequence of chain complexes:

$$0 \to RFC^{(a,b)}(H,h) \overset{i}{\hookrightarrow} RFC^{(a,c)}(H,h) \overset{\pi}{\twoheadrightarrow} RFC^{(b,c)}(H,h) \to 0. \tag{37}$$

As ∂^F reduces the action, it descends to the truncated chain complexes, yielding the truncated homology groups

$$RFH^{<b}(H,h) \qquad \text{and} \qquad RFH^{(a,b)}(H,h).$$

For $\varepsilon > 0$ smaller then the smallest period of a closed Reeb orbit on Σ we define

$$RFH^{(0^{\pm},\infty)}(H,h) := RFH^{(\pm\varepsilon,\infty)}(H,h)$$
$$RFH^{(-\infty,0^{\pm})}(H,h) := RFH^{(-\infty,\pm\varepsilon)}(H,h).$$

The short exact sequence (37) gives the following long exact sequence in homology:

$$\to RFH^{(b,c)}(H,h) \to RFH^{(a,b)}(H,h) \overset{i_*}{\to} RFH^{(a,c)}(H,h) \overset{\pi_*}{\to} RFH^{(b,c)}(H,h) \to \tag{38}$$

If there is no closed Reeb orbit with period in (a,b), then this long exact sequence shows that $RFH^{(b,c)}(H,h)$ and $RFH^{(a,c)}(H,h)$ are isomorphic. This proves that the above definition of $RFH^{(0^{\pm},\infty)}(H,h)$ and $RFH^{(-\infty,0^{\pm})}(H,h)$ is independent from ε, if ε is small enough.
The maps π_* and i_* give $RFH^{(a,b)}$ the structure of a bidirected system. In Theorem 81, we show that

$$RFH^{<b}(H,h) \cong \varprojlim_{a} RFH^{(a,b)}(H,h) \text{ as } a \to -\infty$$

and

$$RFH(H,h) \cong \varinjlim_{b} RFH^{<b}(H,h) \quad \text{as } b \to \infty.$$

The truncated homology groups are independent of J, h and g_h. However, they do depend on the chosen contact form and are therefore in general not invariant under homotopies H_s, except for the groups $RFH^{(0^{\pm},\infty)}(H,h)$ and $RFH^{(-\infty,0^{\pm})}(H,h)$, as we shall see below. For an arbitrary action window we have only the following result.

Corollary 54. *Let $a, b \in (\mathbb{R} \cup \{-\infty,\infty\}) \setminus spec\,(\Sigma,\lambda)$. If H_+ and H_- are two defining Hamiltonians for Σ, then $RFH^{(a,b)}(H_+,h_+)$ and $RFH^{(a,b)}(H_-,h_-)$ are isomorphic.*

Proof: The proof is based on the following idea. If the action spectrum $\mathcal{A}^{H_s}$ is fixed, we can split the homotopy H_s in slower homotopies $H_s^{N,j}$ which allow us to deduce from Corollary 52 that no $\mathcal{A}^{H_s^{N,j}}$-gradient trajectory can cross the action boundaries a and b.

To start, recall that the space of defining Hamiltonians for Σ is convex (Proposition 11). Hence, we can find a homotopy H_s, $0 \leq s \leq 1$, between H_- and H_+, where all H_s are defining Hamiltonians for Σ. For $N \in \mathbb{N}$, we split H_s as in (36) into the slower homotopies

$$H_s^{N,j} := H_{(j+s)/N} \quad \text{for } \ 0 \leq s \leq 1 \qquad \text{and} \qquad H_s^{N,j} := H_{\beta(s)}^{N,j} \quad \text{for } \ s \in \mathbb{R}.$$

Write again $H_j = H_0^{N,j} = H_1^{N,j-1}$ for the ends of these homotopies. Note that $crit\big(\mathcal{A}^{H_j}\big)$ does not depend on H_j but only on Σ as it is for every j the set of all closed Reeb-orbits on Σ. It follows that the action spectrum is $spec\,(\Sigma,\lambda)$ for all j.

As $a, b \notin spec\,(\Sigma, \lambda)$, we can choose $k > 1$ so large such that for any $w \in crit\big(\mathcal{A}^{H_j}\big) = crit\big(\mathcal{A}^H\big)$ with $a < \mathcal{A}^{H_j}(w) < b$ holds

$$a < \frac{k-1}{k} \cdot \mathcal{A}^{H_j}(w) < b \qquad \text{and} \qquad a < \frac{k}{k-1} \cdot \mathcal{A}^{H_j}(w) < b \qquad \forall\ 0 \le j \le N. \quad (*)$$

Then we may choose N so large, such that

$$d := \left(c + \frac{||H^{N,j}||_\infty}{\varepsilon^2}\right) \cdot ||\beta'||_\infty \cdot ||\dot{H}^{N,j}||_\infty$$

is so small, that $kd/(1 - kd - d)$ is smaller then the minimal period of a closed Reeb orbit on Σ. Consider $(v^+, \eta^+) \in RFC(H_{j+1})$, $(v^-, \eta^-) \in RFC(H_j)$ and assume that there exists an $\mathcal{A}^{H_s^{N,j}}$-gradient trajectory connecting them. Assume further that $\mathcal{A}^{H_{j+1}}(v^+, \eta^+) < b$. Corollary 52 then implies that

- if $\mathcal{A}^{H_{j+1}}(v^+, \eta^+) > 0$, then $\mathcal{A}^{H_j}(v^-, \eta^-) \le \frac{k}{k-1} \cdot \mathcal{A}^{H_{j+1}}(v^+, \eta^+)$ and due to $(*)$ therefore $\mathcal{A}_{H_j}(v^-, \eta^-) < b$,
- if $\mathcal{A}^{H_{j+1}}(v^+, \eta^+) < 0$, then $\mathcal{A}^{H_j}(v^-, \eta^-) \le \frac{k-1}{k} \cdot \mathcal{A}^{H_{j+1}}(v^+, \eta^+)$ and again with $(*)$ that $\mathcal{A}^{H_j}(v^-, \eta^-) < b$,
- if $\mathcal{A}^{H_{j+1}}(v^+, \eta^+) = 0$, then $\mathcal{A}^{H_j}(v^-, \eta^-) \le 0 = \mathcal{A}^{H_{j+1}}(v^+, \eta^+) < b$, as otherwise Corollary 52(a) would imply that $\mathcal{A}^{H_{j+1}}(v^+, \eta^+)$ is positive.

We obtain an analog result for a instead of b. Let $\phi^j : RFC(H_{j+1}) \to RFC(H_j)$ be defined as in (35) by counting solutions of the s-dependent Rabinowitz-Floer equation. The above estimates then show that the ϕ^j descend to well-defined maps

$$\phi^j : RFC^{(a,b)}(H_{j+1}) \to RFC^{(a,b)}(H_j),$$

which induce, as in the untruncated case, isomorphisms in homology

$$\Phi^j : RFH^{(a,b)}(H_{j+1}) \to RFH^{(a,b)}(H_j).$$

The composition of the Φ^j then yields $RFH^{(a,b)}(H_+) \cong RFH^{(a,b)}(H_-)$. □

In the definition of the Rabinowitz-Floer homology we assumed that the ambient manifold V is the completion of a Liouville domain. For a Liouville domain W with $\partial W = \Sigma$, we can always set $V := \hat{W}$. The following theorem shows that the Rabinowitz-Floer homology actually only "sees" this completion. This means that we get the same homology if we take larger ambient manifolds $V \supset \hat{W}$.

Theorem 55 (Cieliebak, Frauenfelder & Oancea,[16]).
The Rabinowitz-Floer homology $RFH(W, \Sigma)$ does not depend on the ambient manifold (V, λ), but only on the compact Liouville domain $(W, \lambda|_W)$ bounded by Σ.

Proof: As (V, λ) is the completion of a Liouville domain, the Liouville vector field X_λ is complete. Its flow defines therefore a symplectic embedding $i : \Sigma \times \mathbb{R} \hookrightarrow V$ of the symplectization of (Σ, α) such that $i^*\lambda = e^r \cdot \alpha$ (see Discussion 5).

Pick a cylindrical almost complex structure J_Σ on $\Sigma \times \mathbb{R}$. By Gromov's Monotonicity Lemma (see [49], Prop. 4.3.1 and [42], Lem. 1), there exists an $\varepsilon > 0$ such that J_Σ-holomorphic curves in $\Sigma \times \mathbb{R}$ which meet the level $\Sigma \times \{\log 3\}$ and exit $\Sigma \times [\log 2, \log 4]$ have symplectic area at least ε. Rescaling by $R > 1$, it follows that J_Σ-holomorphic curves which meet the level $\Sigma \times \{\log 3R\}$ and exit the set $\Sigma \times [\log 2R, \log 4R]$ have symplectic area at least $R\varepsilon$.
Now fix a defining Hamiltonian H for Σ. Then there exists a constant $c > 0$ such that H is constant outside $W \cup i\big(\Sigma \times (-\infty, c)\big)$. For any $\log R > c$ pick an admissible almost complex structures J on (V, λ) such that $i^*J = J_\Sigma$ over $\Sigma \times [\log 2R, \log 4R]$. Assume that (v, η) is an $\mathcal{A}^H$-gradient trajectory with asymptotics $(v^\pm, \eta^\pm) \in crit\big(\mathcal{A}^H\big)$ such that v meets the level $i(\Sigma \times \{\log 3R\})$. As the $v^\pm$ are contained in $\Sigma = i(\Sigma \times \{0\})$, v exits the set $i(\Sigma \times [\log 2R, \log 4R])$. Let $U \subset \mathbb{R} \times S^1$ be a connected component of $v^{-1}\big(i(\Sigma \times [\log 2R, \log 4R])\big)$. As X_H vanishes over $i(\Sigma \times [\log 2R, \log 4R])$, it follows that v has over U a symplectic area of at least $R\varepsilon$. This allows us to estimate

$$\mathcal{A}^H(v^+, \eta^+) - \mathcal{A}^H(v^-, \eta^-) = \int_{-\infty}^{\infty} \big|\big|\nabla \mathcal{A}^H(v, \eta)(s)\big|\big|^2 ds \geq \int_U \left|\frac{d}{ds} v\right|^2 dsdt = \int_U v^* d\lambda \geq R\varepsilon.$$

Thus, v can leave $W \cup i(\sigma \times (-\infty, \log 2R))$ only if the action difference of its asymptotics is greater or equal to $R\varepsilon$. By choosing R large enough, we find hence that the moduli space $\widehat{M}(c^-, c^+)$ involves only $\mathcal{A}^H$-gradient trajectories which are contained in the completion $(\hat{W}, \hat{\lambda})$. This shows that $RFH^{(a,b)}(V, \Sigma)$ can be computed using only the completion $(\hat{W}, \hat{\lambda})$ and is therefore independent from the ambient manifold.
Since $RFH(V, \Sigma) = \lim\limits_{b \to \infty} \lim\limits_{-\infty \leftarrow a} RFH^{(a,b)}(V, \Sigma)$ by Theorem 81, the independence carries over to the full Rabinowitz-Floer homology. □

Corollary 56. *The Rabinowitz-Floer homology $RFH(W, \Sigma)$ is invariant under Liouville isomorphisms. It is thus an invariant of the exact contact filling (W, Σ, ξ).*

Proof: At first, we consider only the trivial Liouville isomorphism of (W, λ) with itself. Fix any defining Hamiltonian H_0 for Σ and a Morse function h_0 for $crit\big(\mathcal{A}^{H_0}\big)$. Let $f \in C^\infty(\Sigma)$ be an arbitrary smooth function and consider the exact contact hypersurface $\Sigma^f := \{(y, f(y)) \,|\, y \in \Sigma\}$ in $\hat{W}$. Fix a defining Hamiltonian H_1 for Σ^f and a Morse function h_1 for $crit\big(\mathcal{A}^{H_1}\big)$.
Due to Proposition 11, there exists a homotopy of defining Hamiltonians H_s, $0 \leq s \leq 1$ between H_0 and H_1. It follows from Theorem 53 that $RFH(H_0, h_0)$ and $RFH(H_1, h_1)$ are isomorphic. Thus, the Rabinowitz-Floer homology is invariant under trivial Liouville isomorphisms.

Now consider any Liouville isomorphism $\varphi : (W_0, \Sigma_0, \lambda_0) \to (W_1, \Sigma_1, \lambda_1)$. Fix any defining Hamiltonians H_0, H_1 for Σ_0, Σ_1 and Morse functions h_0, h_1 for $crit\big(\mathcal{A}^{H_0}\big)$ and $crit\big(\mathcal{A}^{H_1}\big)$. It follows from Proposition 8 that there exists an $R > 0$ such that on $\Sigma_0 \times [R, \infty)$ holds $\varphi^*\hat{\lambda}_2 = \hat{\lambda}_1$ and φ is of the form

$$\varphi(y, r) = \big(\psi(y), r - f(y)\big),$$

where $\psi : \Sigma_0 \to \Sigma_1$ is a contact isomorphism satisfying $\psi^*\alpha_1 = e^f \cdot \alpha_0$ for $f \in C^\infty(\Sigma_0)$. The image $\varphi(\Sigma_0 \times \{R\}) \subset \hat{W}_1$ is hence the hypersurface Σ_1^{R-f}.

Pick a defining Hamiltonian H for Σ_1^{R-f} and a Morse function h for $crit\big(\mathcal{A}^H\big)$. It follows from the discussion above, that $RFH(H,h)$ and $RFH(H_1,h_1)$ are isomorphic. Let $\varphi^*H := H\circ\varphi$ and $\varphi^*h = h\circ\varphi$ be the pullbacks. As φ is a global symplectomorphism, we find that φ^*H is a defining Hamiltonian for $\Sigma_0\times\{R\}$.
For any generic almost complex structure J_1 on $\hat{W}_1$, we choose the almost complex structure J_0 on $\hat{W}_0$ to be

$$J_0 = \varphi^*J_1 := D\varphi^{-1}\circ J_1\circ D\varphi.$$

Analogously, let $g_0 = \varphi^*g_1$, be the pullback of a generic metric on $crit\big(\mathcal{A}^{H_1}\big)$. Then, we find that φ^*h is a Morse function on $crit\big(\mathcal{A}^{\varphi^*H}\big)$ and all trajectories with cascades of (φ^*H,φ^*h) are in one-to-one correspondence to the trajectories with cascades of (H,h). Therefore, we have that $RFH(H,h)$ and $RFH(\varphi^*H,\varphi^*h)$ are isomorphic.
A discussion similar to the one above shows that $RFH(H_0,h_0)$ and $RFH(\varphi^*H,\varphi^*h)$ are also isomorphic. Combining the 3 isomorphisms gives $RFH(H_0,h_0)\cong RFH(H_1,h_1)$. $\square$

Definition 57. *Let (W,Σ,λ) be a Liouville domain and let $f:\mathbb{R}\to\mathbb{R}$ be a strictly increasing function with $\lim\limits_{x\to\infty} f(x)=\infty$. Hence, f is invertible and $\lim\limits_{x\to\infty} f^{-1}(x)=\infty$. For $a>0$, let $d^+(\Sigma,a)$ be the $\mathbb{Z}_2$-dimension of the image $i_*\big(RFH^{(0,a)}(W,\Sigma)\big)$ in $RFH^{(0,\infty)}(W,\Sigma)$ and let $d^-(\Sigma,a)$ be the $\mathbb{Z}_2$-dimension of $\pi_*\big(RFH^{(-\infty,0)}(W,\Sigma)\big)$ in $RFH^{(-a,0)}(W,\Sigma)$. Clearly $d^+(\Sigma,a)$ and $d^-(\Sigma,a)$ are increasing functions in a. We define the positive/negative* ***growth rates*** *of class f of a Liouville domain (W,Σ,λ) by*

$$\Gamma^\pm(W,\Sigma,f) := \overline{\lim_{a\to\infty}}\,\frac{\big(f^{-1}\circ\log\big)\big(d^\pm(\Sigma,a)\big)}{\log(a)}\in\{-\infty\}\cup[0,\infty].$$

Remark. For $f=id$, we say that $\Gamma^\pm(W,\Sigma,id)$ is the polynomial growth, for $f=\log$ the logarithmic, for $f=e^x$ the exponential growth.

Corollary 58. *Let (W,Σ,λ) be a Liouville domain. The following growth rates and truncated groups are invariant under Liouville isomorphisms*

$$RFH^{(-\infty,0^\pm)}(W,\Sigma),\quad RFH^{(0^\pm,\infty)}(W,\Sigma)\quad \text{and}\quad \Gamma^\pm(W,\Sigma,f).$$

Remark. As mentioned above, the truncated Rabinowitz-Floer groups $RFH^{(a,b)}(W,\Sigma)$ are in general not invariant under Liouville isomorphisms. This is easy to see, simply rescale Σ, i.e. consider in the symplectization $\Sigma\times\{R\}\hookrightarrow\hat{W}$ for $R\neq 1$.

Proof:
Step 1: Invariance of $RFH^{(-\infty,0^\pm)}(W,\Sigma)$ and $RFH^{(0^\pm,\infty)}(W,\Sigma)$

Following the same arguments as the previous corollary, it suffices to show invariance under homotopies of defining Hamiltonians. Let H_s, $0\le s\le 1$, be a homotopy of Hamiltonians which are defining for exact contact hypersurfaces $\Sigma_s := H_s^{-1}(0)$. For any $N\in\mathbb{N}$, we may split H_s again as is (36) into slower homotopies

$$H_s^{N,j} := H_{(j+s)/N}\quad\text{for }\ 0\le s\le 1\qquad\text{and}\qquad H_s^{N,j} := H^{N,j}_{\beta(s)}\quad\text{for }\ s\in\mathbb{R}$$

and we write as before $H_j = H_0^{N,j} = H_1^{N,j-1}$ for the ends of the homotopies $H_s^{N,j}$.

Fix an $k > 1$. Then we can choose N so large, such that

$$d := \left(c + \frac{||H^{N,j}||_\infty}{\varepsilon^2} \right) \cdot ||\beta'||_\infty \cdot ||\dot{H}^{N,j}||_\infty$$

becomes so small that $kd/(1 - kd - d)$ is smaller then the smallest (positive) minimal period of a closed Reeb orbit on any Σ_s.
Note that we cannot require that closed Reeb orbits stay in a fixed action window as in $(*)$ in Corollary 54, since $spec(\Sigma_s, \lambda)$ is no longer independent from s, except for the common spectral value 0.
Let $(v^+, \eta^+) \in RFC(H_{j+1})$, $(v^-, \eta^-) \in RFC(H_j)$ and assume that there exists an $\mathcal{A}^{H_s^{N,j}}$-gradient trajectory connecting them. For any $a > 0$ we find that our choice of $kd/(1 - kd - d)$ together with Corollary 52 implies that if

(1) $\mathcal{A}^{H_+}(v^+, \eta^+) < a$, then $\mathcal{A}^{H_-}(v^-, \eta^-) < \frac{k}{k-1} \cdot a$,

(2) $\mathcal{A}^{H_+}(v^+, \eta^+) < -\frac{k}{k-1} \cdot a$, then $\mathcal{A}^{H_-}(v^-, \eta^-) < -a$,

(3) $\mathcal{A}^{H_+}(v^+, \eta^+) \leq 0$, then $\mathcal{A}^{H_-}(v^-, \eta^-) \leq 0$.

Let $\phi^j : RFC(H_{j+1}) \to RFC(H_j)$ be defined as in (35) by counting solutions of the s-dependent Rabinowitz-Floer equation. Abbreviating $C := \frac{k}{k-1}$, the statements (1)-(3) show that the ϕ^j descend to well-defined maps

$$\begin{aligned} \phi^j :& \quad RFC^{(0^\pm, a)}(H_{j+1}) \to RFC^{(0^\pm, Ca)}(H_j) \\ \phi^j :& \quad RFC^{(-Ca, 0^\pm)}(H_{j+1}) \to RFC^{(-a, 0^\pm)}(H_j), \end{aligned}$$

which then induce maps in homology

$$\begin{aligned} \Phi^j :& \quad RFH^{(0^\pm, a)}(H_{j+1}) \to RFH^{(0^\pm, Ca)}(H_j) \\ \Phi^j :& \quad RFH^{(-Ca, 0^\pm)}(H_{j+1}) \to RFH^{(-a, 0^\pm)}(H_j). \end{aligned} \tag{39}$$

Considering the inverse homotopy $\overline{H}_s := H_{1-s}$ yields maps

$$\begin{aligned} \Psi^j :& \quad RFH^{(0^\pm, a)}(H_j) \to RFH^{(0^\pm, Ca)}(H_{j+1}) \\ \Psi^j :& \quad RFH^{(-Ca, 0^\pm)}(H_j) \to RFH^{(-a, 0^\pm)}(H_{j+1}). \end{aligned} \tag{40}$$

The compositions maps

$$\begin{aligned} \Psi^j \circ \Phi^j :& \quad RFH^{(0^\pm, a)}(H_{j+1}) \to RFH^{(0^\pm, C^2 \cdot a)}(H_{j+1}) \\ \Psi^j \circ \Phi^j :& \quad RFH^{(-C^2 a, 0^\pm)}(H_{j+1}) \to RFH^{(-a, 0^\pm)}(H_{j+1}) \end{aligned}$$

are just the truncation maps from the long exact sequence (38) induced by the inclusion $RFC^{(0^\pm, a)} \hookrightarrow RFC^{(0^\pm, C^2 \cdot a)}$ and the projection $RFC^{(-C^2 \cdot a, 0^\pm)} \twoheadrightarrow RFC^{(-a, 0^\pm)}$. This holds true as the untruncated maps $\Psi^j \circ \Phi^j$ are isomorphisms. With $a = \infty$, we find that

$$\begin{aligned} \Phi^j :& \quad RFH^{(0^\pm, \infty)}(H_{j+1}) \to RFH^{(0^\pm, \infty)}(H_j) \\ \Phi^j :& \quad RFH^{(-\infty, 0^\pm)}(H_{j+1}) \to RFH^{(-\infty, 0^\pm)}(H_j) \end{aligned}$$

are isomorphisms, as $\Phi^j \circ \Psi^j$ and $\Psi^j \circ \Phi^j$ are isomorphisms. Combining all Φ^j yields $RFH^{(0^\pm, \infty)}(H_0) \cong RFH^{(0^\pm, \infty)}(H_1)$ and $RFH^{(-\infty, 0^\pm)}(H_0) \cong RFH^{(-\infty, 0^\pm)}(H_1)$.

Step 2: Invariance of $\Gamma^{\pm}(W,\Sigma,f)$

For any $a<\infty$, we find that the composition of all Φ^j resp. all Ψ^j yields maps

$$\begin{aligned}
\Phi^+ :&\quad RFH^{(0,a)}(H_1) \to RFH^{(0,D\cdot a)}(H_0)\\
\Phi^- :&\quad RFH^{(-D\cdot a,0)}(H_1) \to RFH^{(-a,0)}(H_0)\\
\Psi^+ :&\quad RFH^{(0,a)}(H_0) \to RFH^{(0,D\cdot a)}(H_1)\\
\Psi^- :&\quad RFH^{(-D\cdot a,0)}(H_0) \to RFH^{(-a,0)}(H_1),
\end{aligned}$$

with $D := C^N = (\frac{k}{k-1})^N$. Their compositions yield again natural truncation maps:

$$\begin{aligned}
\Psi^+\circ\Phi^+ :&\quad RFH^{(0,a)}(H_1) \to RFH^{(0,D^2\cdot a)}(H_1),\\
\Psi^-\circ\Phi^- :&\quad RFH^{(-D^2\cdot a,0)}(H_1) \to RFH^{(-a,0)}(H_1).
\end{aligned}$$

Therefore, we get the following ladder-shaped diagrams:

$$\begin{tikzcd}
\cdots & \cdots \\
RFH^{(0,D^4\cdot a)}(H_0) \arrow[u] \arrow[r] & RFH^{(0,D^5\cdot a)}(H_1) \arrow[u] \arrow[ul] \\
RFH^{(0,D^2\cdot a)}(H_0) \arrow[u] \arrow[r] & RFH^{(0,D^3\cdot a)}(H_1) \arrow[u] \arrow[ul] \\
RFH^{(0,a)}(H_0) \arrow[u] \arrow[r] & RFH^{(0,Da)}(H_1) \arrow[u] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots \arrow[d] \arrow[dr] & \cdots \arrow[d] \\
RFH^{(-D^4\cdot a,0)}(H_0) \arrow[d] \arrow[dr] & RFH^{(-D^5\cdot a,0)}(H_1) \arrow[l] \arrow[d] \\
RFH^{(-D^2\cdot a,0)}(H_0) \arrow[d] \arrow[dr] & RFH^{(-D^3\cdot a,0)}(H_1) \arrow[l] \arrow[d] \\
RFH^{(-a,0)}(H_0) & RFH^{(-Da,0)}(H_1). \arrow[l]
\end{tikzcd}$$

They imply the following chain of inequalities:

$$d^{\pm}(H_0,a) \leq d^{\pm}(H_1,Da) \leq d^{\pm}(H_0,D^2a) \leq \ldots$$

Now assume that $RFH^{(0,\infty)}(H_0)$ and $RFH^{(-\infty,0)}(H_0)$ are infinite dimensional. As these are obtained by direct/inverse limits, this implies that $d^{\pm}(H_0,a)\to\infty$ as $a\to\infty$, which yields in particular

$$\lim_{a\to\infty}\frac{\log(D)}{f^{-1}(\log(d^{\pm}(H_0,a)))} = 0.$$

With this result, we obtain

$$\begin{aligned}
\frac{1}{\Gamma^{\pm}(H_0,f)} = \lim_{a\to\infty}\frac{\log(a)}{f^{-1}(\log(d^{\pm}(H_0,a)))} &= \lim_{a\to\infty}\frac{\log(Da)}{f^{-1}(\log(d^{\pm}(H_0,a)))}\\
&\geq \lim_{a\to\infty}\frac{\log(Da)}{f^{-1}(\log(d^{\pm}(H_1,Da)))} = \frac{1}{\Gamma^{\pm}(H_1,f)}.
\end{aligned}$$

The same argument works in the opposite direction, so that we get altogether $\Gamma^{\pm}(H_0,f)=\Gamma^{\pm}(H_1,f)$. In the remaining case, where $RFH^{(0,\infty)}(H_0)$ or $RFH^{(-\infty,0)}(H_0)$ are finite dimensional, the growth rate is either zero, if $0<\dim RFH^{(0,\infty)}(H_0)<\infty$, or $-\infty$, if $0=\dim RFH^{(0,\infty)}(H_0)$. $\square$

3.3. The Conley-Zehnder index

To obtain more information about its structure, we endow the Rabinowitz-Floer homology in the next section with a $\mathbb{Z}$-grading via the Conley-Zehnder index μ_{CZ}, just as in regular Floer homology. To define μ_{CZ}, let $Sp(2n)$ denote the group of $2n \times 2n$ symplectic matrices. In [18], Conley and Zehnder introduced a Maslov type index for paths $\Psi : [0,1] \to Sp(2n)$. Their index assigns an integer $\mu_{CZ}(\Psi)$ to every path Ψ, provided that $\Psi(0) = \mathbb{1}$ and $\det(\mathbb{1} - \Psi(1)) \neq 0$. Later, Robbin and Salamon gave in [43] a different definition, thus extending μ_{CZ} to arbitrary paths. It goes as follows:
Any smooth path $\Psi : [a,b] \to Sp(2n)$ can be expressed as a solution of an ordinary differential equation

$$\dot{\Psi}(t) = J_0 S(t) \Psi(t), \qquad \Psi(a) \in Sp(2n),$$

where $t \mapsto S(t) = S(t)^T$ is a smooth path of symmetric matrices and J_0 the standard almost complex structure. A time $t \in [a,b]$ is called a crossing if $\det(\mathbb{1} - \Psi(t)) = 0$. The crossing form at a crossing t is a quadratic form $\Gamma(\Psi, t)$ defined for $\xi_0 \in \ker(\mathbb{1} - \Psi(t))$ by the formula[4]

$$\Gamma(\Psi, t)\xi_0 = \langle \xi_0, S(t)\xi_0 \rangle. \tag{$*$}$$

A crossing t is called regular, if $\Gamma(\Psi, t)$ is non-degenerate. Regular crossings are isolated. For a path Ψ with only regular crossings, the Conley-Zehnder index is defined by

$$\mu_{CZ}(\Psi; a, b) := \frac{1}{2}\operatorname{sign}\Gamma(\Psi, a) + \sum_{a<t<b} \operatorname{sign}\Gamma(\Psi, t) + \frac{1}{2}\operatorname{sign}\Gamma(\Psi, b), \tag{41}$$

where the sum runs over all crossings $t \in (a,b)$. Here, $\operatorname{sign} A$ denotes the signature of A, i.e. the number of positive eigenvalues minus the number of negative eigenvalues. Note that $\Gamma(\Psi, a) = 0$ or $\Gamma(\Psi, b) = 0$ if a or b are not crossings. To ease notation, we will often omit one or both boundaries if they are clear from the context.
The index μ_{CZ} has (among others) the following properties:

(Naturality) For any path $\Phi : [a,b] \to Sp(2n)$ holds $\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi)$

(Homotopy) $\mu_{CZ}(\Psi_s)$ is constant in s for any homotopy Ψ_s with fixed endpoints

(Product) If $Sp(2n) \oplus Sp(2n')$ is identified with a subgroup of $Sp(2(n+n'))$ in the obvious way, then $\mu_{CZ}(\Psi \oplus \Psi') = \mu_{CZ}(\Psi) + \mu_{CZ}(\Psi')$.

The homotopy property allows us to define $\mu_{CZ}(\Psi; a, b)$ also for paths with non-regular crossings, provided its ends a and b are regular crossings or no crossings. Just perturb Ψ through a homotopy Ψ_s, $0 \leq s \leq 1$ to a path Ψ_1 with only regular crossings and set $\mu_{CZ}(\Psi; a, b) := \mu_{CZ}(\Psi_1; a, b)$.
Next, we calculate the indices $\mu_{CZ}(\Psi)$ of some explicit paths.

[4] Actually, $(*)$ cannot be found in [43]. However, it follows from their Rem. 5.4 together with their Thm. 1.1(2) applied to the Lagrangian frame $(Id, \Psi)^T$ for $Graph(\Psi)$.

Lemma 59. *Let $\Psi_1, \Psi_2, \Psi_3 : [0,T] \to Sp(2)$ be the following paths:*

$$\Psi_1(t) = e^{it}, \quad \Psi_2(t) = e^{-it}, \quad \Psi_3(t) = \begin{pmatrix} e^{\rho(t)} & 0 \\ 0 & e^{-\rho(t)} \end{pmatrix}, \ \rho \in C^1(\mathbb{R}).$$

Then, their Conley-Zehnder indices are given as follows:

$$\begin{aligned} \mu_{CZ}(\Psi_1) &= \left\lfloor \frac{T}{2\pi} \right\rfloor + \left\lceil \frac{T}{2\pi} \right\rceil &&= \begin{cases} \frac{T}{\pi} & \text{if } T \in 2\pi\mathbb{Z} \\ 2\left\lfloor \frac{T}{2\pi} \right\rfloor + 1 & \text{otherwise,} \end{cases} \\ \mu_{CZ}(\Psi_2) &= \left\lfloor \frac{-T}{2\pi} \right\rfloor + \left\lceil \frac{-T}{2\pi} \right\rceil &&= -\mu_{CZ}(\Psi_1), \\ \mu_{CZ}(\Psi_3) &= 0. \end{aligned}$$

Proof: We note that $\Psi_1(t)$ is represented by the real matrix

$$\Psi_1(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad \text{with} \qquad i \mathrel{\widehat{=}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = J_0.$$

Hence we calculate that

$$\dot{\Psi}_1(t) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix} = J_0 \circ S_1(t) \circ \Psi_1(t),$$

where $S_1(t) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The crossings are exactly the set $2\pi\mathbb{Z} \cap [0,T]$ and the crossing form is always $\Gamma(\Psi_1, t) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ having signature 2. Hence

$$\mu_{CZ}(\Psi_1) = \begin{cases} 1 + 2\left\lfloor \frac{T}{2\pi} \right\rfloor & \text{if } T \notin 2\pi\mathbb{Z} \\ 1 + 2\left\lfloor \frac{T}{2\pi} \right\rfloor - 1 = \frac{T}{\pi} & \text{if } T \in 2\pi\mathbb{Z}. \end{cases}$$

The formula for Ψ_2 is completely analog with $S_2(t) = -\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For Ψ_3 we calculate

$$\dot{\Psi}_3(t) = \begin{pmatrix} \dot{\rho}(t)e^{\rho(t)} & 0 \\ 0 & -\dot{\rho}(t)e^{-\rho(t)} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dot{\rho}(t) \\ -\dot{\rho}(t) & 0 \end{pmatrix} \begin{pmatrix} e^{\rho(t)} & 0 \\ 0 & e^{-\rho(t)} \end{pmatrix}$$

and hence $S_3(t) = -\left(\begin{smallmatrix} 0 & \dot{\rho}(t) \\ \dot{\rho}(t) & 0 \end{smallmatrix}\right)$. But this matrix has signature 0 as its eigenvalues are $\pm\dot{\rho}(t)$. It follows that $\operatorname{sign} \Gamma(\Psi_3, t) = 0$ for every crossing t and thus $\mu_{CZ}(\Psi_3) = 0$. $\square$

In [47], Salamon and Zehnder introduced yet another approach to μ_{CZ}. They showed that there is a continuous extension $\rho : Sp(2n) \to S^1$ of the determinant map $\det : U(n) = Sp(2n) \cap O(2n) \to S^1$, which is unique when one requires some additional properties. Moreover, they showed that the space $Sp(2n)^*$ of symplectic matrices not having 1 as eigenvalue has two connected components which are semi-simple connected in $Sp(2n)$.
Now, any path $\Psi : [0,T] \to Sp(2n)$ with $\Psi(0) = \mathbb{1}$ and $\Psi(T) \in Sp(2n)^*$ admits a homotopic unique extension $\Psi : [0, T+1] \to Sp(2n)$ such that $\Psi|_{[T,T+1]}$ connects $\Psi(T)$ in

$Sp(2n)^*$ to one of the matrices $W^+ = -\mathbb{1}$ or $W^- = diag(2, -1, \ldots, -1, 1/2, -1, \ldots, -1)$. To define $\mu_{CZ}(\Psi)$, choose a lift $\alpha : [0, T+1] \to \mathbb{R}$ of $\rho \circ \Psi : [0, T+1] \to S^1$ and set

$$\mu_{CZ}(\Psi; 0, T) = \mu_{CZ}(\Psi, T) := \frac{\alpha(T+1) - \alpha(0)}{\pi}.$$

That both definitions of $\mu_{CZ}(\Psi)$ coincide for paths Ψ with $\Psi(0) = \mathbb{1}$ and $\Psi(T) \in Sp(2n)^*$ is shown in [43]. The advantage of the second approach is that it allows us to define the mean index $\Delta(\Psi, T)$ of a path $\Psi : [0, T] \to Sp(2n)$ as

$$\Delta(\Psi, T) := \frac{\alpha(T) - \alpha(0)}{\pi}.$$

Note that $\Delta(\Psi, T)$ is in general not an integer and that the definition of $\Delta(\Psi, T)$ does not require that $\Psi(T) \in Sp(2n)^*$.
The mean index allows us to estimate μ_{CZ} for iterated paths.

Lemma 60 (**Iterations formula**). *Assume that $\Psi : [0, T] \to Sp(2n)$, $\Psi(0) = \mathbb{1}$ is an iterated path, i.e. it holds that $S(t + \tau) = S(t)$ for some $\tau \in \mathbb{R}$ and $\dot{\Psi}(t) = J_0 S(t)\Psi(t)$ as above. Equivalently, we could require that*

$$\Psi(k\tau + t) = \Psi(t)\Psi(\tau)^k, \qquad \textit{for any } k \in \mathbb{Z}.$$

Under these conditions, we have that

$$\mu_{CZ}(\Psi, k\tau) = k \cdot \Delta(\Psi, \tau) + R \qquad \textit{with } |R| \le 2n.$$

Proof: Without loss of generality, we assume that Ψ has only regular crossings and that $(k\tau, k\tau + t]$ contains no crossings for t small enough. We define for $\Psi(T) \in Sp(2n)^*$

$$r(\Psi, T) := \mu_{CZ}(\Psi, T) - \Delta(\Psi, T).$$

Note that r is continuous on $(k\tau, k\tau + t]$, as $\mu_{CZ}(\Psi, T)$ is constant for $T \in (k\tau, k\tau + t]$ and $\Delta(\Psi, T)$ is continuous in T. Moreover, it follows from (41) that

$$\begin{aligned}\mu_{CZ}(\Psi, k\tau) &= \mu_{CZ}(\Psi, k\tau + t) - \frac{1}{2}\Gamma(\Psi, k\tau)\\ &= \Delta(\Psi, k\tau + t) + r(\Psi, k\tau + t) - \frac{1}{2}\Gamma(\Psi, k\tau)\\ &= \Delta(\Psi, k\tau) + \lim_{t \searrow 0} r(\Psi, k\tau + t) - \frac{1}{2}\Gamma(\Psi, k\tau).\end{aligned}$$

It is shown in [47], Lem. 3.4, that $|r(\Psi, k\tau + t)| < n$. That $\Delta(\Psi, k\tau) = k \cdot \Delta(\Psi, \tau)$ follows from the fact that $\Delta(\Psi, k\tau)$ is the winding number of the path $\rho \circ \Psi|_{[0,k\tau]}$, which is the k-fold iteration of the path $\rho \circ \Psi|_{[0,\tau]}$. As $\Gamma(\Psi, k\tau)$ is the signature of an $m \times m$ matrix with $m \le 2n$, we have also $|\Gamma(\Psi, k\tau)| \le 2n$. Hence, it follows that

$$\mu_{CZ}(\Psi, k\tau) = k \cdot \Delta(\Psi, \tau) + R,$$

where $$|R| = \left| \lim_{t \searrow 0} r(\Psi, k\tau + t) - \frac{1}{2}\Gamma(\Psi, k\tau) \right| \le n + n = 2n. \qquad \square$$

3.4. A $\mathbb{Z}$-grading for RFH

Now, we associate to each contractible periodic Reeb trajectory v a Conley-Zehnder index $\mu_{CZ}(v)$, which allows us to define a $\mathbb{Z}$-grading for Rabinowitz-Floer homology. For simplicity, let us make the following assumptions:

(A) *The map $i_* : \pi_1(\Sigma) \to \pi_1(W)$ induced by the inclusion is injective.*

(B) *The integral $I_{c_1} : \pi_2(W) \to \mathbb{Z}$ of the first Chern class $c_1(TW)$ vanishes on spheres.*

Remark.

- Assumption (A) is automatically satisfied if Σ is simply connected.
- One can associate a Conley-Zehnder index to every closed Reeb trajectory v, but it will depend on the trivialization of $v^*\xi$. In order to fix the trivialization, we restrict ourself to contractible trajectories. These have, due to (B), a unique Conley-Zehnder index (see below). Note that the two ends of a solution of the Rabinowitz-Floer equation are either both contractible or not. Hence, the contractible closed Reeb trajectories generate a subcomplex of $RFC(W, \Sigma)$. Its homology is by abuse of notation also denoted by $RFH(W, \Sigma)$. If Σ is simply connected this version of Rabinowitz-Floer homology coincides with the original one.

To obtain the (transversal) Conley-Zehnder index of a closed contractible Reeb trajectory v choose a map u from the unit disc $D \subset \mathbb{C}$ to Σ such that $u(e^{2\pi i t}) = v(t)$. The existence of such maps is guaranteed by assumption (A). Now choose a symplectic trivialization $\Phi : D \times \mathbb{R}^{2n-2} \to u^*\xi$ of the pullback bundle $(u^*\xi, u^*d\alpha)$. Such trivializations exist and are homotopically unique as D is contractible. The linearization of the Reeb flow ψ^t along v with respect to Φ defines a path Ψ in the group $Sp(2n-2)$ starting at $\mathbb{1}$ by

$$\Psi(t) := \Phi(v(t))^{-1} \circ d\psi^t(v(0)) \circ \Phi(v(0)).$$

The Conley-Zehnder index of this path is the (transverse) Conley-Zehnder index $\mu_{CZ}(v)$. It is independent from the choice of u due to assumption (B). Indeed, if we choose another disc $u' : D \to \Sigma$ with $u'|_{\partial D} = v$ and another trivialization Φ' of $u'^*\xi$, then we can glue u and u' together to a map $w : S^2 \to \Sigma$.
Now, $(w^*\xi, w^*d\alpha)$ is trivial. Indeed, $\int w^*c_1(\xi) = \int w^*c_1(TV) = 0$ and hence $[w^*c_1(\xi)] = 0 \in H^2(S^2)$, as the complement of ξ in TV is trivialized by the Reeb and Liouville vector field. Therefore, there exists a trivialization Φ'' of $w^*\xi$ and both Φ and Φ' are homotopic to Φ'' along v (see [47, Lem.5.2]).

The transversal Conley-Zehnder index μ_{CZ} allows us to grade RFH as follows.

Proposition 61 (**Frauenfelder & Cieliebak, [14]**). *If $\pi_1(\Sigma) \to \pi_1(V)$ is injective and I_{c_1} vanishes, then we have a $\mathbb{Z}$-grading of the Rabinowitz-Floer homology $RFH(\Sigma, V)$, which is independent of V and given by the index*

$$\mu(c) := \mu_{CZ}(c) + ind_h(c) - \frac{1}{2}\dim_c\big(crit(\mathcal{A}^H)\big) + \frac{1}{2},$$

where $c \in crit(h)$ and $\dim_c\big(crit(\mathcal{A}^H)\big)$ is the real dimension of the connected component of $crit\big(\mathcal{A}^H\big)$ which contains c.

Proof: Recall that the boundary operator ∂^F counted points in the zero-dimensional manifolds $\mathcal{M}(c^-,c^+,m)$, which where given as quotients $\widehat{\mathcal{M}}(c^-,c^+,m)\,/_{\mathbb{R}^m}$ if $m\neq 0$ and $\widehat{\mathcal{M}}(c^-,c^+,0)\,/_{\mathbb{R}}$ if $m=0$. The Global Transversality Theorem 38 gave the following dimension formula near a flow line (v,η) with m cascades by

$$\begin{aligned}\dim_{(v,t)}\ \widehat{\mathcal{M}}(c^-,c^+,m) &\overset{m\neq 0}{=} \Big(\mu_{CZ}(c^+,\bar{c}^+)+ind_h(c^+)-\frac{1}{2}\dim_{c^+}(crit(\mathcal{A}^H))\Big)\\ &\qquad -\Big(\mu_{CZ}(c^-,\bar{c}^-)+ind_h(c^-)-\frac{1}{2}\dim_{c^-}(crit(\mathcal{A}^H))\Big)\\ &\qquad + m-1+\sum_{k=1}^{m}2c_1(\bar{v}_k^-\#v_k\#\bar{v}_k^+)\\ \dim_{(v,t)}\ \widehat{\mathcal{M}}(c^-,c^+,m) &\overset{m=0}{=} ind_h(c^+)-ind_h(c^-).\end{aligned}$$

Using condition (B), we get a dimension formula for the moduli space by

$$\begin{aligned}\dim_{(v,t)}\ \mathcal{M}(c^-,c^+,m) = &\Big(\mu_{CZ}(c^+,\bar{c}^+)+ind_h(c^+)-\frac{1}{2}\dim_{c^+}(crit(\mathcal{A}^H))\Big)\\ &-\Big(\mu_{CZ}(c^-,\bar{c}^-)+ind_h(c^-)-\frac{1}{2}\dim_{c^-}(crit(\mathcal{A}^H))\Big)-1\\ =\ &\mu(c^+)-\mu(c^-)-1.\end{aligned}$$

Note that the formula holds also for $m=0$, as then $\mu_{CZ}(c^+,\bar{c}^+)=\mu_{CZ}(c^-,\bar{c}^-)$ and $\dim_{c^+}\big(crit(\mathcal{A}^H)\big)=\dim_{c^-}\big(crit(\mathcal{A}^H)\big)$, since (v,η) is simply a Morse trajectory on one connected component of $crit(\mathcal{A}^H)$. The moduli space is hence zero-dimensional if and only if $\mu(c^+)-\mu(c^-)=1$. It follows that ∂^F reduces the index μ exactly by 1 so that μ provides a well-defined grading for the homology. □

Discussion 62. The term $\frac{1}{2}$ in the definition of μ, which does not appear in [14], has no influence on the relative grading given by the other terms. It normalizes μ such that it takes values in $\mathbb{Z}$ and fits with the grading of symplectic (co)homology (see [16]). Note that this convention differs also from the one used previously by the author in [23], where $\frac{1}{2}\dim\Sigma$ was added instead of $\frac{1}{2}$. So all indices in this work are shifted by $-\frac{1}{2}\dim\Sigma+\frac{1}{2}=-n+1=-(n-2)-1$ in comparison to the indices in [23].

The grading μ allows us to define more refined invariants for contact structure:

Definition 63. *Let $c_k(W):=\dim_2 RFH_k(W,\Sigma)$ denote the k^{th} Betti-number of the Rabinowitz-Floer homology of the filling W of Σ. We define*

$$C_k(\Sigma):=\{c_k(W)\,|\,W \text{ Liouville domain, } \partial W=\Sigma\}\subset[0,\infty]$$

to be the set of all Betti numbers $c_k(W)$ for varying fillings W of Σ.

Observe that $C_k(\Sigma)$ is (trivially) independent from W. It is therefore an invariant of (Σ,ξ), as $RFH_k(W,\Sigma)$ was apart from ξ only dependent on W. The most simplest case is, when $|C_k(\Sigma)|=1$, i.e. if the group $RFH_k(W,\Sigma)$ does not depend on W at all and is itself an invariant of the contact structure.

Proposition 64. *Assume that* (W, Σ) *is a Liouville domain satisfying (A) and (B). Then* $RFH_k(W, \Sigma)$ *is independent of* W*, if* Σ *admits a contact form for which the closed contractible Reeb orbits are Morse-Bott (MB) and for all* $c_* = (v_*, \eta_*) \in RFC_*(W, \Sigma)$ *with* $* \in \{k-1, k, k+1\}$ *holds that* $\eta_{k-1} \geq \eta_k \geq \eta_{k+1}$.

Proof: At first, observe that $RFH(W, \Sigma)$ does not depend on the particular contact form. Moreover, the grading μ of the chain complex and the chain complex itself do not depend on W. The chain groups $RFC_k(W, \Sigma)$ are hence independent of W. Now consider $(v_*, \eta_*) \in RFC_*(W, \Sigma)$ with $* \in \{k-1, k, k+1\}$.
If $\eta_{k-1} \geq \eta_k \geq \eta_{k+1}$, the Lemmas 15 and 16 tell us that if there are flow lines with cascades from c_{k-1} to c_k or from c_k to c_{k+1} then they must have zero cascades, i.e. they are h-Morse flow lines on C_k, as each cascade would reduce the action. Since h-Morse flow lines are independent of the filling W, so are the boundary operators

$$\partial_*^F : RFC_{*+1}(W, \Sigma) \to RFC_*(W, \Sigma), \quad * \in \{k-1, k\}$$

and hence the quotient

$$RFH_k(W, \Sigma) = \frac{\ker \partial_{k-1}^F}{\operatorname{im} \partial_k^F}. \qquad \square$$

In [16, Cor.1.15], Cieliebak, Frauenfelder and Oancea proved another criterion under which $RFH_k(W, \Sigma)$ does not depend on W, i.e. where $|C_k(\Sigma)| = 1$ for all k.

Theorem 65. *Let* (W, Σ) *be a Liouville domain,* $\dim W = 2n$*, satisfying assumption (A) and (B).* $RFH_k(W, \Sigma)$ *is independent of* W *for all* k *if* Σ *admits a contact form for which all contractible closed Reeb orbits* v *are Morse-Bott (MB) and satisfy*

$$\mu_{CZ}(v) > 3 - n.$$

Proof: *(Sketch)*
The theorem is shown by proving that all $\mathcal{A}^H$-gradient trajectories lie entirely in the symplectization of Σ and are hence independent of the filling. This follows from a Gromov compactness argument, as trajectories leaving the symplectization would lead to the bubbling-off of holomorphic planes. The latter cannot happen due to the condition $\mu_{CZ}(v) > 3 - n$. $\square$

We finish the section with a short discussion of what happens with (A) and (B) under handle attachment. Let (W, Σ) be a Liouville domain satisfying (A) and (B). Assume that (W', Σ') is obtained from (W, Σ) by attaching a k-handle H_k^{2n} as described in Section 5. In general, one cannot (topologically) expect that (A) still holds for (W', Σ') as is shown by attaching a 1-handle to the unit ball in $\mathbb{R}^3$. The result is diffeomorphic to the full 2-torus, which violates (A). However, we have the following lemma.

Lemma 66. *Assume that* (W, Σ) *satisfies (A),* $\dim W = 2n \geq 4$ *and that* (W', Σ') *is obtained by attaching a* k*-handle* H_k^{2n}*,* $k \leq n-2$*. Then* (W', Σ') *satisfies (A) if*

1. $k \geq 3$ *or*
2. $k = 1$*,* W *has 2 components* (W_1, Σ_1)*,* (W_2, Σ_2) *and* H_k^{2n} *is glued to* W *so that* Σ' *is the connected sum* $\Sigma_1 \# \Sigma_2$ *resp.* W' *is the boundary connected sum* $W_1 \# W_2$.

Proof:

@1. The first part of this proof follows Milnor, [39, Lem.2]. Set $l = 2n - k$ and note that $k < l$. Let X denote the space which is obtained by gluing the handle $H_k^{2n} \cong D^k \times D^l$ to Σ. It is formed from the topological sum $\Sigma + (D^k \times D^l)$ by identifying $S^{k-1} \times D^l$ with the attaching region in Σ. The subset $\Sigma \cup (D^k \times 0)$ is a deformation retract of X. This subset is formed from Σ by attaching a k-cell. It follows thus that the map $\pi_1(\Sigma) \to \pi_1(X)$ induced by inclusion is an isomorphism as $k \geq 3$. But Σ' is also embedded topologically in X and similar arguments show that $\pi_1(\Sigma') \to \pi_1(X)$ is also an isomorphism as $l > k \geq 3$.
Analogously, we consider the space Y which is obtained by gluing the handle H_k^{2n} to W. Note that $Y = W'$ and the same reasonings as for Σ shows that $\pi_1(W) \to \pi_1(W')$ is also an isomorphism. Combining all these spaces, we obtain the following diagram:

$$\begin{tikzcd} \pi_1(\Sigma) \arrow[r] \arrow[d, "(a)"] & \pi_1(X) \arrow[d, "(b)"] & \pi_1(\Sigma') \arrow[l] \arrow[d, "(c)"] \\ \pi_1(W) \arrow[r] & \pi_1(W') \arrow[r, equal] & \pi_1(W') \end{tikzcd} .$$

As all the maps are induced by inclusions, the above diagram commutes. Moreover, all horizontal maps are isomorphisms. As (a) is injective by assumption, it follows from the commutativity of the left square that (b) is also injective. Then it follows from the commutativity of the right square that (c) is also injective.

@2. Without loss of generality we assume that $W = W_1 \cup W_2$ and $\Sigma = \Sigma_1 \cup \Sigma_2$, such that $W' = W_1 \# W_2$ and $\Sigma' = \Sigma_1 \# \Sigma_2$. Now we apply the van Kampen Theorem in Σ' to the two sets $\Sigma_1' \cong \Sigma_1 \setminus \{pt.\}$ and $\Sigma_2' \cong \Sigma_2 \setminus \{pt.\}$ coming from Σ_1 and Σ_2 plus the connecting cylinder. This gives a map $\pi_1(\Sigma_1 \setminus \{pt.\}) * \pi_1(\Sigma_2 \setminus \{pt.\}) \to \pi_1(\Sigma_1 \# \Sigma_2)$ induced by inclusion.
Analogously, we obtain a map $\pi_1(W_1 \setminus \{pt.\}) * \pi_1(W_2 \setminus \{pt.\}) \to \pi_1(W_1 \# W_2)$. As the intersections $\Sigma_1' \cap \Sigma_2' \cong D^1 \times S^{2n-2}$ and $W_1' \cap W_2' \cong D^1 \times D^{2n-1}$ are simply connected (as $n \geq 2$), both maps are isomorphisms. Moreover, they fit into the following diagram:

$$\begin{tikzcd} \pi_1(\Sigma_1) * \pi_1(\Sigma_2) \arrow[d, "(a)"] & \pi_1(\Sigma_1 \setminus \{pt.\}) * \pi_1(\Sigma_2 \setminus \{pt.\}) \arrow[l] \arrow[r] \arrow[d, "(b)"] & \pi_1(\Sigma_1 \# \Sigma_2) \arrow[d, "(c)"] \\ \pi_1(W_1) * \pi_1(W_2) & \pi_1(W_1 \setminus \{pt.\}) * \pi_1(W_2 \setminus \{pt.\}) \arrow[l] \arrow[r] & \pi(W_1 \# W_2) \end{tikzcd} .$$

Again all the maps are induced by inclusion, so this diagram also commutes. The two horizontal maps on the left are isomorphisms as $\dim \Sigma, \dim W \geq 3$. Therefore, all horizontal maps are isomorphisms. As (a) is injective by assumption, it follows as above that (c) is also injective. $\square$

A similar result holds for the behaviour of (B) under handle attachment.

Lemma 67. *Let (W, Σ) be as above, satisfying (B). Let (W', Σ') be obtained by attaching a k-handle. Then (W', Σ') also satisfies (B) if $k = 1$ or $k \geq 3$.*

Proof: Assume that $I_{c_1} : \pi_2(W) \to \mathbb{Z}$ vanishes. As W' is obtained from W by attaching a handle, $W \subset W'$ is an open subset and $c_1(TW')|_V = c_1(TW)$. As in the proof above, there is a retraction $\rho : W' = W \cup H_k^{2n} \to W \cup D^k$, where ∂D^k is identified with an embedded sphere S^{k-1} in Σ. Let $w : S^2 \to W'$ be any smooth sphere.

- If $k \geq 3$, we may assume that there is a point q in the interior of the attached disc D^k such that $q \notin im(\rho \circ w)$. Note that there exists a retraction $R : W' \setminus \{q\} \to W$ and that $R \circ w$ is well-defined. It follows that w and $R \circ w$ are homotopic and hence
$$\int_{S^2} w^* c_1(TW') = \int_{S^2} (R \circ w)^* c_1(TW') = \int_{S^2} (R \circ w)^* c_1(TW) = 0.$$

- If $k = 1$, then D^1 is a line. Hence we may split $\rho \circ w$ into a finite number of spheres $\tilde{w}_i$ such that no $\tilde{w}_i$ passes over the line. It may well hit its centre but $D^1 \not\subseteq image(\tilde{w}_i)$. Now, we may homotope each $\tilde{w}_i$ into W to obtain spheres $w_i : S^2 \to W$. Then
$$\begin{aligned}\int_{S^2} w^* c_1(TW') = \int_{S^2} (\rho \circ w)^* c_1(TW') &= \sum_i \int_{S^2} \tilde{w}_i^* c_1(TW') \\ &= \sum_i \int_{S^2} w_i^* c_1(TW) = 0. \qquad \square\end{aligned}$$

4. Some algebra for Floer theory

4.1. Direct and inverse limits

In the previous sections, we gave a description of Rabinowitz-Floer homology with the help of direct and inverse limits. Moreover, the construction of symplectic (co)homology in Section 6 will also need these limits. In this section, we recall the definition of these algebraic limits and prove some of their properties. Our discourse is based on the fundamental books [4] and [21]. Throughout this section let R denote a unitary ring.

Definition 68. *A relation $\alpha \leq \beta$ on a set M is called a **quasi order** if it is reflexive and transitive. A **directed set** $(M, \leq)$ is a quasi ordered set such that for each pair $\alpha, \beta \in M$ there exists a $\gamma \in M$ with $a \leq \gamma$ and $\beta \leq \gamma$. A directed set M' is a **subset** of a directed set $(M, \leq)$ if $M' \subset M$ and the quasi order on M' is the restriction of $\leq$ to $M' \times M'$. A subset M' is **cofinal** in M if for each $\alpha \in M$ exists a $\beta \in M'$ such that $\alpha \leq \beta$.*

Definition 69.

- *A **direct system** (X, ι) of R-modules over a directed set M is a function which attaches to each $\alpha \in M$ an R-module X^α and to each pair $\alpha \leq \beta$ an R-linear map*
$$\iota^{\beta\alpha} : X^\alpha \to X^\beta$$
such that $\qquad \iota^{\alpha\alpha} = id \qquad$ *and* $\qquad \iota^{\gamma\alpha} = \iota^{\gamma\beta}\iota^{\beta\alpha} \qquad \forall \alpha \leq \beta \leq \gamma.$

- *An **inverse system** (X, π) of R-modules over a directed set M is a function which attaches to each $\alpha \in M$ an R-module X_α and to each pair $\alpha \leq \beta$ an R-linear map*
$$\pi_{\alpha\beta} : X_\beta \to X_\alpha$$
such that $\qquad \pi_{\alpha\alpha} = id \qquad$ *and* $\qquad \pi_{\alpha\gamma} = \pi_{\alpha\beta}\pi_{\beta\gamma} \qquad \forall \alpha \leq \beta \leq \gamma.$

Definition 70.

- *Let (X, ι) and (Y, j) be direct systems over M resp. N. An R-**homomorphism** $\Phi : (X, \iota) \to (Y, j)$ consists of an order preserving map $\phi : M \to N$, $\phi(\alpha) = \alpha'$ and R-linear maps $\phi^\alpha : X^\alpha \to Y^{\alpha'}$ for each $\alpha \in M$ such that for $\alpha \leq \beta$ the following diagram commutes*
$$\begin{tikzcd} X^\alpha \arrow[r, "\iota^{\beta\alpha}"] \arrow[d, "\phi^\alpha"'] & X^\beta \arrow[d, "\phi^\beta"] \\ Y^{\alpha'} \arrow[r, "j^{\beta'\alpha'}"] & Y^{\beta'} \end{tikzcd} .$$

- *Let (X, π) and (Y, p) be inverse systems over M resp. N. An R-**homomorphism** $\Phi : (X, \pi) \to (Y, p)$ consists of an order preserving map $\phi : N \to M$, $\phi(\alpha') = \alpha$*

and R-linear maps $\phi_{\alpha'} : X_\alpha \to Y_{\alpha'}$ for each $\alpha' \in N$ such that for $\alpha' \leq \beta'$ the following diagram commutes

$$\begin{tikzcd} X_\alpha \arrow[d, "\phi_{\alpha'}"'] & X_\beta \arrow[l, "\pi_{\alpha\beta}"'] \arrow[d, "\phi_{\beta'}"] \\ Y_{\alpha'} & Y_{\beta'} \arrow[l, "\pi_{\alpha'\beta'}"'] \end{tikzcd} .$$

Definition 71. *Let (X, π) be an inverse system of R-modules. The **inverse limit** X_∞ of (X, π) is the following sub-R-module of the product ΠX_α:*

$$X_\infty := \left\{ x = (x_\alpha) \in \Pi X_\alpha \,\middle|\, \pi_{\alpha\beta}(x_\beta) = x_\alpha \;\forall\, \alpha \leq \beta \right\}.$$

Note that we have for each $\alpha \in M$ a projection

$$\pi_\alpha : X_\infty \to X_\alpha,\ \pi_\alpha(x) = x_\alpha \qquad \text{satisfying} \qquad \pi_\alpha = \pi_{\alpha\beta}\pi_\beta \quad \forall \alpha \leq \beta.$$

These maps allow us to describe X_∞ alternatively by the following universal property: For each R-module Y with a family of R-linear maps $\tau_\alpha : Y \to X_\alpha$ which satisfy $\tau_\alpha = \pi_{\alpha\beta}\tau_\beta$ for all $\alpha \leq \beta$ there exists a unique R-linear map $\tau : Y \to X_\infty$ such that for any $\alpha \in M$ the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, dotted, "\tau"] \arrow[dr, "\tau_\alpha"'] & & X_\infty \arrow[dl, "\pi_\alpha"] \\ & X_\alpha & \end{tikzcd} .$$

Any R-homomorphism $\Phi : (X, \pi) \to (Y, p)$ between two inverse systems induces an R-linear map $\phi_\infty : X_\infty \to Y_\infty$ via $\phi_\infty((x_\alpha)_{\alpha \in M}) = (\phi_{\alpha'}(x_\alpha))_{\alpha' \in N}$. We say that ϕ_∞ is the inverse limit of the ϕ_α.

Definition 72. *Let (X, ι) be a direct system of R-modules. Let $\bigoplus X^\alpha$ denote the direct sum and let $Q \subset \bigoplus X^\alpha$ be the submodule generated by all elements of the from $\iota^{\beta\alpha}(x^\alpha) - x^\alpha$ for any $a \leq \beta$. The **direct limit** of (X, ι) is the quotient module*

$$X^\infty := \bigoplus X^\alpha \big/ Q .$$

Note that the inclusions $X^\alpha \subset \bigoplus X^\alpha$ induce for each $\alpha \in M$ an R-linear map

$$\iota^\alpha : X^\alpha \to X^\infty,\ \iota^\alpha(x^\alpha) = [x^\alpha] \quad \text{satisfying} \quad \iota^\alpha = \iota^\beta \iota^{\beta\alpha} \quad \forall\, \alpha \leq \beta.$$

Again, there is an alternative description of X^∞ by a universal property: For each R-module Y with a family of R-linear maps $\tau^\alpha : X^\alpha \to Y$ satisfying $\tau^\alpha = \tau^\beta \iota^{\beta\alpha}$ for all $\alpha \leq \beta$ there exists a unique R-linear map $\tau : X^\infty \to Y$ such that for any $\alpha \in M$ the following diagram commutes

$$\begin{tikzcd} Y & & X^\infty \arrow[ll, dotted, "\tau"'] \\ & X_\alpha \arrow[ul, "\tau^\alpha"] \arrow[ur, "\iota^\alpha"'] & \end{tikzcd} .$$

Any R-homomorphism $\Phi : (X, \pi) \to (Y, p)$ between direct systems induces an R-linear map $\phi^\infty : X^\infty \to Y$ by $\phi^\infty\big(\big[\sum x^\alpha\big]\big) = \sum \phi^\alpha(x^\alpha)$. We call ϕ^∞ the direct limit of the ϕ^α.

Theorem 73 (**[21], Thm. 5.4.; [4], §6, no3, prop. 4**).

- *The direct limit is an exact functor, that is if*

$$(A,\iota) \xrightarrow{\phi} (B,\iota) \xrightarrow{\psi} (C,\iota)$$

is an exact sequence of direct systems, the following limit sequence is also exact

$$A^\infty \xrightarrow{\phi^\infty} B^\infty \xrightarrow{\psi^\infty} C^\infty.$$

- *The inverse limit is a left exact functor, that is if*

$$0 \longrightarrow (A,\pi) \xrightarrow{\phi} (B,\pi) \xrightarrow{\psi} (C,\pi)$$

is an exact sequence of inverse systems, the following limit sequence is also exact

$$0 \longrightarrow A_\infty \xrightarrow{\phi_\infty} B_\infty \xrightarrow{\psi_\infty} C_\infty.$$

Remark. The inverse limit is in general not an exact functor (see [4] for an example). However, it is exact if R is a field and all R-modules $A_\alpha, B_\alpha, C_\alpha$ have finite dimension.

Theorem 74 (**[21], Thm. 4.13/Cor. 4.14, Thm 3.15/Cor 3.16**). *Let $(X,\pi),(X,\iota)$ be an inverse/direct system over the directed set M_X and let $M_Y \subset M_X$ be a cofinal subset. Let $(Y,\pi),(Y,\iota)$ be the restricted systems over M_Y, i.e. $Y_\alpha = X_\alpha$ resp. $Y^\alpha = X^\alpha$ for all $\alpha \in M_Y \subset M_X$. Then the limits of the systems coincide, i.e.*

$$\varprojlim X_\alpha = \varprojlim Y_\alpha \qquad \text{and} \qquad \varinjlim X^\alpha = \varinjlim Y^\alpha.$$

Proof:

a) <u>inverse limits</u>: Consider the R-linear map

$$\phi : \varprojlim X_\alpha \to \varprojlim Y_\alpha, \qquad (x_\alpha)_{\alpha\in M_X} \mapsto (x_\alpha)_{\alpha\in M_Y},$$

which deletes in the family (x_α) all elements with $\alpha \notin M_Y$. Define a map

$$\psi : \varprojlim Y_\alpha \to \varprojlim X_\alpha,$$

$$(y_\alpha)_{\alpha\in M_Y} \mapsto (x_\alpha)_{\alpha\in M_X}, \qquad x_\alpha := \begin{cases} y_\alpha & \text{if } \alpha \in M_Y \\ \pi_{\alpha\beta}(y_\beta) & \text{if } \alpha \notin M_Y,\ \beta \in M_Y,\ \beta \geq \alpha. \end{cases}$$

As $M_Y \subset M_X$ is cofinal, there exists for any $\alpha \in M_X$ a β such that $\alpha \leq \beta$. As (Y,π) is an inverse system over the directed set M_Y, the definition of x_α does not depend on the particular choice of β. The map ψ is hence well-defined. Obviously, ψ is also R-linear and satisfies $\phi \circ \psi = id|_{X_\infty}$ and $\psi \circ \phi = id|_{Y_\infty}$. In particular, ϕ is an isomorphism and the two limits do therefore coincide.

b) <u>direct limits</u>: Consider the following composition of inclusion and projection:

$$\varinjlim Y^\alpha = \bigoplus Y^\alpha \big/ Q_Y \hookrightarrow \bigoplus X^\alpha \big/ Q_Y \twoheadrightarrow \bigoplus X^\alpha \big/ Q_X = \varinjlim X^\alpha.$$

It yields an R-linear map $\phi : \varinjlim Y^\alpha \to \varinjlim X^\alpha$, $\left[\sum y^\alpha\right]_{Q_Y} \mapsto \left[\sum y^\alpha\right]_{Q_X}$.
Define a map $\psi : \varinjlim X^\alpha \to \varinjlim Y^\alpha$ as follows. Any class $\xi \in \varinjlim X^\alpha$ has a representative $\left[\sum_{i=1}^n x^{\alpha_i}\right]_{Q_X}$ with $x^{\alpha_i} \in X^{\alpha_i}$. As $M_Y \subset M_X$ is cofinal, we may choose $\beta \in M_Y$ such that $\beta \geq \alpha_i$ for all i. Then, ξ is also represented by $\left[\sum_{i=1}^n \iota^{\beta\alpha_i}(x^{\alpha_i})\right]_{Q_X}$. Now we set

$$\psi(\xi) = \psi\left(\left[\textstyle\sum_{i=1}^n x^{\alpha_i}\right]_{Q_X}\right) = \left[\textstyle\sum_{i=1}^n \iota^{\beta\alpha_i}(x^{\alpha_i})\right]_{Q_Y}.$$

It is not to difficult to see that this definition does not depend on the particular choice of β nor on the choice of the representatives for ξ. The map ψ is hence well-defined. Obviously, ψ is also an R-linear map and satisfies $\phi \circ \psi = id|_{Y^\infty}$ and $\psi \circ \phi = id|_{X^\infty}$. In particular, ϕ is an isomorphism and the limits do coincide. $\square$

Remark. Note that any direct/inverse limit of R-homomorphisms does also only depend on cofinal subsets.

Definition 75. *A* ***bidirect system*** (X, π, ι) *of R-modules over two directed sets M and N consists of a function which attaches to each pair $(\alpha, \beta) \in M \times N$ an R-module X_α^β, to each triple $(\alpha_1, \alpha_2, \beta) \in M^2 \times N$ with $\alpha_1 \leq \alpha_2$ an R-homomorphism*

$$\pi_{\alpha_1\alpha_2}^\beta : X_{\alpha_2}^\beta \to X_{\alpha_1}^\beta,$$

to each triple $(\alpha, \beta_1, \beta_2) \in M \times N^2$ with $\beta_1 \leq \beta_2$ an R-homomorphism

$$\iota_\alpha^{\beta_2\beta_1} : X_\alpha^{\beta_1} \to X_\alpha^{\beta_2}$$

such that for each fixed β the tuple (X^β, π^β) is an inverse system over M and for each fixed α the tuple (X_α, ι_α) is a direct system over N. Moreover, for each quadruple $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$ the following diagram has to commute:

$$\begin{tikzcd}
X_{\alpha_2}^{\beta_1} \arrow[r, "\pi_{\alpha_1\alpha_2}^{\beta_1}"] \arrow[d, "\iota_{\alpha_2}^{\beta_2\beta_1}"'] & X_{\alpha_1}^{\beta_1} \arrow[d, "\iota_{\alpha_1}^{\beta_2\beta_1}"] \\
X_{\alpha_2}^{\beta_2} \arrow[r, "\pi_{\alpha_1\alpha_2}^{\beta_2}"] & X_{\alpha_1}^{\beta_2}
\end{tikzcd}.$$

It follows that (X_∞, ι_∞) is a direct system over N and that (X^∞, π^∞) is an inverse system over M. Note that in general the limits of these two systems are not equal. In other words direct and inverse limit do not commute, i.e.

$$\varinjlim \varprojlim X \neq \varprojlim \varinjlim X.$$

4.2. Abstract Floer theory

In this section, we present an axiomatic model for Morse and Floer theory over possibly non-compact manifolds. We follow [15], who claim that it all goes back to a suggestion made by D. Salamon. The Main Theorem 81 states that the two basic approaches in this setup - working over a Novikov completion or taking limits - yields the same homology, i.e. that $FH \cong \varinjlim \varprojlim FH^{[a,b]}$. Our proof of this crucial theorem is less abstract (compared to [15], though perhaps tedious) and completely selfcontained.

Definition 76. *A Floer triple over a ring R is a tuple $\mathcal{F} = (C, f, m)$ consisting of a set C, a function $f : C \to \mathbb{R}$ and a function $m : C \times C \to R$ such that the following conditions hold.*

i. The set $C_a^b := \{c \in C \,|\, a \leq f(c) \leq b\}$ is finite for each $-\infty < a \leq b < \infty$.

ii. If $m(c_1, c_2) \neq 0$ then $f(c_1) \leq f(c_2)$.

iii. For all $c_1, c_3 \in C$ holds $\displaystyle\sum_{c_2 \in C} m(c_1, c_2) \cdot m(c_2, c_3) = 0$.

Remark.

- Assertions *i.* and *ii.* assure that the sum in *iii.* is actually finite.
- We call f the action function, call the elements of C critical points of action $f(c)$ and call C_a^b the set of critical points in the action window $[a, b]$. The value $m(c_1, c_2)$ represents the count (with signs) of Floer cylinders with asymptotics c_1 and c_2.
- The condition $f(c_1) \leq f(c_2)$ in assertion *ii,* reflects the fact that we also include Morse-Bott theories. For pure Morse theories replace "$\leq$" by "$<$".
- Assertion *i.* implies that the action spectrum *spec*$(f) := f(C) \subset \mathbb{R}$ is closed and discrete. We can hence find a monotone injective map from $f(C)$ to $\mathbb{Z}$. With the help of this map we will henceforth assume that in fact

  $$f(C) \subset \mathbb{Z}.$$

  This is solely for notational reasons and does not effect the generality of our theorems. Under this assumption, the first assertion takes the following form

  i'. For every $a \in \mathbb{Z}$, the set $C_a^a = f^{-1}(a) = \{c \in C \,|\, f(c) = a\}$ is finite.
- Note the changed notation for the action window in this chapter. In all other chapters we write $C^{(a,b)}$ instead of C_a^b. In particular, the lower index denotes the lower end of the action window and not the grading with respect to the boundary operator, which plays only a minor role in this chapter.
- Assertion *iii.* guarantees that the formula $dc_0 := \sum_{c \in C} m(c, c_0) \cdot c$ will define a boundary operator.

Definition 77. *For $a \leq b \in \mathbb{Z}$ let $FC_a^b := C_a^b \otimes R$ be the free R-module generated by C_a^b. For any $a \leq c \leq b$ let*

$$\pi_{ca}^b : FC_a^b \longrightarrow FC_c^b \cong FC_a^b \Big/ FC_a^{c-1} \qquad \text{and} \qquad \iota_a^{bc} : FC_a^c \longrightarrow FC_a^b \cong FC_a^c \oplus FC_{c+1}^b$$

be the natural projection resp. inclusion.

Obviously, (FC_a^b, π, ι) is a bidirect system over $\mathbb{Z}\times\mathbb{Z}$, where the quasi order in the upper index is "$\leq$", while it is "$\geq$" on the lower one.

Definition 78. *We set* $\quad FC^b := \varprojlim_a FC_a^b \qquad$ *and* $\qquad FC := \varinjlim_b FC^b = \varinjlim_b \varprojlim_a FC_a^b.$

Note that $s \in FC^b$ can be interpreted as a formal sum $\quad s = \sum_{a\leq b} s_a,\ s_a \in FC_a^a.$

Then, $s \in FC$ is also interpreted as a formal sum $\quad s = \sum_{a\in\mathbb{Z}} s_a,\ s_a \in FC_a^a.$

As FC is obtained by a direct sum, we find for the second sum that there exists a $b \in \mathbb{Z}$ such that $s_a = 0$ for all $a > b$. Therefore, we write the elements of FC also as $s = \sum_{a\leq b} s_a$. Note that the maps π and ι extend to the limits and we have in particular

$$\pi_c^b \ : \ FC^b \to FC_c^b, \qquad \sum_{a\leq b} s_a \mapsto \sum_{a=c}^{b} s_a.$$

We call the summands s_a in the representation of $s \in FC$ the ***a-th coefficient*** of s and we use for the A-th coefficient of s the notation

$$K_A(s) = K_A\Big(\sum_{a\leq b} s_a\Big) := s_A.$$

Note that $K_a : FC \to FC_a^a$ as a map is well-defined and linear. The following important fact is an easy consequence of the definition of FC.

Lemma 79. *Any sequence of coefficients $(s_a)_{a\in\mathbb{Z}}$ with $s_a = 0$ for all $a > b$ with some $b \in \mathbb{Z}$ determines uniquely an element in FC.*

Now let us take a strictly decreasing sequence $(s^b)_{b\leq b_0} \subset FC$, $s^b = \sum_{a\leq b} s_a^b \in FC^b$. We associate to the formal Laurent series $\sum_{b\leq b_0} s^b$ a value in FC by the well-known formula

$$\sum_{b\leq b_0} s^b := \sum_{a\leq b_0} \sum_{b=a}^{b_0} s_a^b = \sum_{a\leq b_0} \sum_{b=a}^{b_0} K_a(s^b).$$

Observe that the inner sum is finite thus determining an element $s_a \in FC_a^a$. The value of the infinite sum $\sum s^b$ is then the unique element in FC represented by the sequence of coefficients $(s_a)_{a\leq b_0}$. The well-known features of this infinite sum are summarized in the following lemma.

Lemma 80. *For $b_0 \in \mathbb{Z}$ and $b \le b_0$ let $s^b, t^b \in FC^b$. Then*

(a) $\displaystyle\sum_{b\le b_0} s^b = s^{b_0} + s^{b_0-1} + \cdots + s^{b_0-k+1} + \sum_{b\le b_0-k} s^b$ *(associativity)*

(b) $\displaystyle\sum_{b\le b_0}(s^b + t^b) = \sum_{b\le b_0} s^b + \sum_{b\le b_0} t^b$ *(countable associativity)*

(c) $\displaystyle s^{b_0} = \sum_{b\le b_0}(s^b - s^{b-1})$ *(telescope sum)*

(d) If $d : FC \to FC$ is a homomorphism with $d(FC^b) \subset FC^b$, then

$$d\Big(\sum_{b\le b_0} s^b\Big) = \sum_{b\le b_0} d(s^b). \qquad \textit{(countable linearity)}$$

Proof: We calculate

(a)
$$K_A\Big(\sum_{b\le b_0} s^b\Big) = K_A\Big(\sum_{a\le b_0}\sum_{b=a}^{b_0} s_a^b\Big) = \sum_{b=A}^{b_0} s_A^b$$

$$K_A\Big(s^{b_0} + \cdots + s^{b_0-k+1} + \sum_{b\le b_0-k} s^b\Big) = K_A\big(s^{b_0}\big) + \cdots + K_A\big(s^{b_0-k+1}\big) + K_A\Big(\sum_{b\le b_0-k} s^b\Big)$$
$$= s_A^{b_0} + \cdots + s_A^{b_0-k+1} + \sum_{b=A}^{b_0-k} s_A^b = \sum_{b=A}^{b_0} s_A^b$$

So all coefficients of the left side of equation (a) coincide with all coefficients of the right side. Hence they represent the same element in FC. The proof of the subsequent assertions follows the same scheme.

(b)
$$K_A\Big(\sum_{b\le b_0}(s^b + t^b)\Big) = K_A\Big(\sum_{a\le b_0}\sum_{b=a}^{b_0}(s_a^b + t_a^b)\Big) = \sum_{b=A}^{b_0}(s_A^b + t_A^b)$$

$$K_A\Big(\sum_{b\le b_0} s^b + \sum_{b\le b_0} t^b\Big) = K_A\Big(\sum_{b\le b_0} s^b\Big) + K_A\Big(\sum_{b\le b_0} t^b\Big) = \sum_{b=A}^{b_0} s_A^b + \sum_{b=A}^{b_0} t_A^b$$

(c)
$$\sum_{b\le b_0}(s^b - s^{b-1}) \overset{(b)}{=} \sum_{b\le b_0} s^b - \sum_{b\le b_0-1} s^b \overset{(a)}{=} s^{b_0} + \sum_{b_0-1} s^b - \sum_{b_0-1} s^b = s^{b_0}$$

(d)
$$K_A\Big[d\Big(\sum_{b\le b_0} s^b\Big)\Big] \overset{(a)}{=} K_A\Big[d\Big(s^{b_0} + \cdots + s^A + \sum_{b\le A-1} s^b\Big)\Big]$$
$$= K_A\Big[d(s^{b_0}) + \cdots + d(s^A) + d\Big(\sum_{b\le A} s^b\Big)\Big]$$
$$\overset{(*)}{=} K_A\big(d(s^{b_0})\big) + \cdots + K_A\big(d(s^A)\big) = \sum_{b=A}^{b_0} K_A\big(d(s^b)\big)$$

where $(*)$ holds as $d\big(FC^{A-1}\big) \subset FC^{A-1}$ and hence $K_A\big(d\big(\sum_{b\le A-1} s^b\big)\big) = 0$.

On the other hand $K_A\Big(\sum_{b\le b_0} d(s^b)\Big) = K_A\Big(\sum_{a\le b_0}\sum_{b=a}^{b_0} K_a\big(d(s^b)\big)\Big) = \sum_{b=A}^{b_0} K_A\big(d(s^b)\big).$ $\square$

We define a boundary operator d on FC_a^b as the linear extension of

$$dx := \sum_{y\in C_a^b} m(y,x)\cdot y, \qquad x \in C_a^b.$$

Note that assertion iii. implies that $d^2 = 0$, i.e. that d really is a boundary operator. As d does not increase action (due to ii.), we find that d satisfies

$$\pi_{ca}^b \circ d = d \circ \pi_{ca}^b \qquad \text{and} \qquad \iota_a^{bc} \circ d = d \circ \iota_a^{bc} \qquad \forall\ a \le c \le b. \tag{42}$$

Therefore, d induces maps on FC^b and FC, which we will also denote with d. Explicitly, they are given as the infinite linear extension of

$$dx := \sum_{y\in C} m(y,x)\cdot y, \qquad x \in C.$$

Note that ii. and iii. imply that $d^2 = 0$ and that (42) still holds, now including $a = -\infty$ and $b = \infty$. Note that we may express the boundary operator on FC_a^b as $\pi_a^b \circ d$, where d is the operator on FC. We denote the resulting homologies by

$$FH_a^b, \quad FH^b \quad \text{and} \quad FH.$$

The elements of these homologies are denoted by $[s]_a^b$, $[s]^b$ and $[s]$, where we suppress the upper index whenever it is clear from the context.
As a consequence of (42), we find that π and ι descend to linear maps in homology, still denoted by π and ι:

$$\pi_{ca}^b\ :\ FH_a^b \to FH_c^b \qquad \iota_a^{bc}\ :\ FH_a^c \to FH_a^b \qquad \forall -\infty \le a \le c \le b \le \infty.$$

As (FC_a^b, π, ι) resp. (FC^b, ι) are bidirect resp. direct systems, it follows that (FH_a^b, π, ι) resp. (FH^b, ι) are bidirect resp. direct systems as well. Observe that we have in particular for every fixed b_0 and any $a \le b_0$ the following map

$$\pi_a^{b_0}\ :\ FH^{b_0} \to FH_a^{b_0}, \qquad [s] \mapsto \Big[\sum_{b=a}^{b_0} s_b\Big]_a, \qquad \text{where} \qquad s = \sum_{b\le b_0} s_b \in FC^{b_0},$$

which satisfies $\pi_c^{b_0} = \pi_{ca}^{b_0}\pi_a^{b_0}$ for every $a \le c$. Hence, there exists by the universal property a unique R-linear map

$$\phi\ :\ FH^{b_0} \to \varprojlim_a FH_a^{b_0}, \qquad [s] \mapsto \left(\Big[\sum_{b=a}^{b_0} s_b\Big]_a\right)_{a\le b_0}.$$

Analogously, we have for every $b \in \mathbb{Z}$ a map

$$\iota^b : FH^b \to FH,$$

which satisfies $\iota^a = \iota^b \iota^{ba}$ for every $a \leq b$. By the universal property, we hence obtain a unique R-linear map

$$\psi : \varinjlim_b FH^b \to FH, \qquad \Big[\sum_{i=1}^{n} \sigma_i\Big] \mapsto \sum_{i=1}^{n} \iota^{b_i}(\sigma_i), \qquad \sigma_i \in FH^{b_i}.$$

Theorem 81.

a) The map ϕ is always surjective. If R is a field, it is injective and yields for every $b \in \mathbb{Z}$ an isomorphism

$$FH^b \cong \varprojlim_a FH_a^b.$$

b) The map ψ is an isomorphism for any R, i.e. $\quad FH \cong \varinjlim_b FH^b.$

If R is a field, we have therefore $\quad FH \cong \varinjlim_b \varprojlim_a FH_a^b.$

In the notation of the other sections, this reads as $\quad FH \cong \varinjlim_b \varprojlim_a FH^{(a,b)}.$

Proof:

(a) claim: ϕ is surjective.
proof: Let $(\sigma_a)_{a\leq b} \in \varprojlim FH_a^b$ be arbitrary with $\sigma_a \in FH_a^b$ and $\pi_{ca}\sigma_a = \sigma_c$ for all $a \leq c \leq b$. We will construct inductively coefficients $s_a \in FC_a^a$ such that for all $A \leq b$ holds $\sigma_A = \big[\sum_{a=A}^b s_a\big]_A$. Then we set $s := \sum_{a\leq b} s_a \in FC$ and calculate

$$ds = d\sum_{a\leq b} s_a = \sum_{a\leq b} ds_a = \sum_{A\leq b}\sum_{a=A}^{b} K_A(ds_a) = \sum_{A\leq b} K_A\Big(d\sum_{a=A}^{b} s_a\Big) = 0$$

as $\big[\sum_{a=A}^b s_a\big]_A = \sigma_A$ and hence $\pi_A\big(d\sum_{a=A}^b s_a\big) = 0$ in FC_A^b. Thus, we have that s is a cycle and so that $[s] \in FH^b$ well-defined. We then see that ϕ is surjective, as

$$\phi\big([s]\big) = \Bigg(\Big[\sum_{a=A}^{b} s_a\Big]_A\Bigg)_{A\leq b} = \big(\sigma_A\big)_{A\leq b}.$$

For the construction of the s_a first consider the following short exact sequence

$$0 \to FC_a^a \xrightarrow{\iota} FC_a^b \xrightarrow{\pi} FC_{a+1}^b \to 0,$$

which yields in homology the long exact sequence

$$\dots \to FH_{a+1}^b \xrightarrow{\delta} FH_a^a \xrightarrow{\iota} FH_a^b \xrightarrow{\pi} FH_{a+1}^b \xrightarrow{\delta} FH_a^a \to \dots \tag{43}$$

Here, the connecting homomorphism δ is given by

$$\delta[s]_{a+1}^b := \big[\pi_a ds\big]_a^a,$$

which is the class of the a-th coefficient of ds, as $ds \in FC^a$.

Here and in the following we suppress the indices and write ι or π instead of ι_a^{ba} or $\pi_{a+1,a}^b$, whenever it is clear from the context.
Let $s_b \in FC_b^b$ be any representative of σ_b, i.e. $\sigma_b = [s_b]_b$. Now assume that $s_{b-1}, \dots, s_{A+1}$ have already been constructed such that $\sigma_{A+1} = \big[\sum_{a=A+1}^b s_a\big]_{A+1}$. We apply (43) with $a = A$ and find $\sigma_{A+1} = \pi\sigma_A \in im(\pi) = \ker(\delta)$. Hence there exists a $n_A \in FC_A^A$ such that $\pi_A\big(d(\sum_{a=A+1}^b s_a) - dn_A\big) = 0 \in FC_A^b$. Thus, we find that $\big[\sum_{a=A+1}^b s_a - n_A\big]_A \in FH_A^b$ is well defined. As $\pi\big[\sum_{a=A+1}^b s_a - n_A\big]_A = \big[\sum_{a=A+1}^b s_a\big]_{A+1} = \sigma_{A+1} = \pi\sigma_A$ we find that $\sigma_A - \big[\sum_{a=A+1}^b s_a - n_A\big]_A \in \ker(\pi) = im(\iota)$. Hence there exists a $\tilde{s}_A \in FC_A^A$ such that $\sigma_A = \big[\sum_{a=A+1}^b s_a - n_A + \tilde{s}_A\big]_A$. With $s_A := \tilde{s}_A - n_A$ we obtain $\sigma_A = \big[\sum_{a=A}^b s_a]_A$.

claim: ϕ is injective if R is a field.
proof: Assume that $\phi\big([s]\big) = 0 \in \varprojlim FH_a^b$ for some $s = \sum_{a\le b} s_a \in FC^b$. We will construct a sequence of coefficients $n_a \in FC_a^a$ such that for all $A \le b$ holds

$$\pi_A\Big(s - d\sum_{a=A}^b n_a\Big) = 0 \qquad (\text{in } FC_A^b),$$

which is equivalent to $s - d\sum_{a=A}^b n_a \in FC^{A-1}$.
Then we set $\beta := \sum_{a\le b} n_a$ and we find that

$$s - d\beta \in FC^a\ \forall a \le b \quad \Rightarrow \quad s - d\beta \in \bigcap_{a\le b} FC^a = \{0\} \quad \Rightarrow \quad s = d\beta.$$

It follows that $[s] = 0 \in FH^b$ and therefore that ϕ is injective.
To start, note that $\phi\big([s]\big) = \big(\big[\sum_{a=A}^b s_a\big]_A\big)_{A\le b} = 0$ is equivalent to

$$\begin{aligned} &\forall A \le b: \quad 0 = \big[\textstyle\sum_{a=A}^b s_a\big]_A \in FH_A^b \\ \Leftrightarrow \quad &\forall A \le b\ \exists \beta_A \in FC_A^b: \qquad s - d\beta_A \in FC^{A-1}. \end{aligned} \tag{$*$}$$

The sequence $n_b, n_{b-1}, \dots$ we are going to construct will actually satisfy the following slightly stronger condition

$$\forall A \le b\ \forall k \ge 0\ \exists \beta_A^k \in FC^A \ : \ s - d\Big[\Big(\sum_{a=A+1}^b n_a\Big) + \beta_A^k\Big] \in FC^{A-k}. \tag{$**$}$$

For the empty sequence, condition $(**)$ has to hold for $A = b$, which is true by $(*)$. Now assume that $n_b, \dots, n_{A+1}$ have already been constructed such that $(**)$ holds. Then let $G_k \subset FC_A^A$ denote the set of all A-coefficients n_A such that there exists a $\beta_k \in FC^{A-1}$ with $s - d\big[\big(\sum_{a=A+1}^b n_a\big) + n_A + \beta_k\big] \in FC^{A-1-k}$. Due to assumption $(**)$, we find that $G_k \neq \emptyset$. Moreover, G_k has the structure of an affine subspace of FC_A^A and $G_{k+1} \subset G_k$. As FC_A^A is a finite dimensional vector space (R is a field and C_A^A finite), we find that the common intersection is not empty, i.e.

$$G := \bigcap_{k\ge 0} G_k \neq \emptyset.$$

Choose any $n_A \in G$. The construction of G shows that $(**)$ holds for $n_b, \dots, n_A$.

(b) claim: ψ is injective.
proof: Assume that $\psi\big(\big[\sum_{i=1}^n \sigma^i\big]\big) = 0 \in FH$ for some $\big[\sum_{i=1}^n \sigma^i\big] \in \varinjlim FH^b$, $\sigma^i \in FH^{b_i}$. Let $s^i \in FC^{b_i}$ be such that $\sigma^i = [s^i]^{b_i}$. It follows that $\psi\big(\big[\sum_{i=1}^n \sigma^i\big]\big) = \big[\sum_{i=1}^n s^i\big] = 0$. Hence, there exists a $\beta \in FC$ such that $\sum_{i=1}^n s^i = d\beta$. Then there exists a $b_0 \in \mathbb{Z}$ such that $b_0 \geq b_i$ for all i and $\beta \in FC^{b_0} \subset FC$. This implies that

$$\sum_{i=1}^n \iota^{b_0 b_i}\sigma^i = \Big[\sum_{i=1}^n s^i\Big]^{b_0} = \big[d\beta\big]^{b_0} = 0 \in FH^{b_0},$$

which shows that $\big[\sum_{i=1}^n \sigma^i\big] = 0 \in \varinjlim FH^b$.

claim: ψ is surjective.
proof: Consider any $[s] \in FH$, $s \in FC$. Then there exists a $b_0 \in \mathbb{Z}$ such that $s \in FC^{b_0}$. As $ds = 0$, we find that s represents a well-defined class $[s]^{b_0} \in FH^{b_0}$. Let $\big[[s]^{b_0}\big]$ denote its class in $\varinjlim FH^b$. Then we see that $\psi\big(\big[[s]^{b_0}\big]\big) = [s]$, thus showing that ψ is surjective. □

4.3. Reducing filtered complexes

Recall from the previous section that abstract Floer theory is based on a set C which generates (countably linear) the chain complex FC and the homology FH. Now we are going to address the question to what extend we can reduce C and still get the same homology FH. Under the assumption that R is a field such that all FH_a^b are vector spaces, we will show that we can replace for any a the set C_a^a by any basis of FH_a^a.
The motivation behind this is the following. Given a Morse-Bott setup, C_a^a is the set of all critical points of a Morse function on the critical manifold $\mathcal{N}^a$ on the action level a, while the groups FH_a^a are the Morse homology of $\mathcal{N}^a$. The fact that we can built FH from FH_a^a then shows that for calculations of FH we may always algebraically pretend that we have a perfect Morse function on $\mathcal{N}^a$, as the critical points of such a function form a basis of FH_a^a. This will be used in Section 7.2 when we calculate the Rabinowitz-Floer homology for some Brieskorn manifolds, where we know the singular homology of the critical manifolds, but we do not know if we have perfect Morse functions.
To be more explicit, let the reduced chain groups be

$$F\mathscr{C} := \varinjlim_b \varprojlim_a \textstyle\bigoplus_{c=a}^b FH_c^c$$

and consider the following variation of the long exact sequence (43):

$$\cdots \to FH_b^b \xrightarrow{\delta} FH^{b-1} \xrightarrow{\iota} FH^b \xrightarrow{\pi} FH_b^b \xrightarrow{\delta} FH^{b-1} \to \ldots, \tag{44}$$

where the map δ is induced by the boundary operator d and given by $\delta[n_b]_b^b = [dn_b]^{b-1}$.
We want to use δ to define a boundary operator on $F\mathscr{C}$. For that, let

$$K := \delta(F\mathscr{C}) \subset \varinjlim \varprojlim \textstyle\bigoplus_{c=a}^b FH^{c-1}.$$

Using the sequence (44), we define below a (non-canonical) map $\Phi : K \to F\mathscr{C}$. Setting $\partial := \Phi \circ \delta$, we will see that $\partial^2 = 0$. The Reduction Theorem 85 then tells us that the homology of $\big(F\mathscr{C}, \partial\big)$ is isomorphic to FH. Here, the isomorphism is given by a map $\Psi : \ker \delta \to FH$. We will use this map to show in Theorem 82 that always (not only over field coefficients) $F\mathscr{C}$ generates FH, i.e that the whole homology cannot have more elements then all the singular homologies of the critical manifolds togehter.
Note that one could take $\Phi = \pi$. However, the resulting homology $(F\mathscr{C}, \partial)$ is then isomorphic to the second page of the spectral sequence of the filtration, which is in general not isomorphic to FH. The reason for this discrepancy is that π discards $\ker \pi = im(\iota)$. Our Φ will also use the $(\ker \pi)$-part of K.

Remark. Let $\mathscr{C}_a^a$ denote a basis of FH_a^a and set $\mathscr{C} = \bigcup \mathscr{C}_a^a$. Then we find that $\mathscr{C}$ generates $F\mathscr{C}$. The isomorphism between $(F\mathscr{C}, \partial)$ and FH then shows that we can replace C by $\mathscr{C}$ and obtain the same homology FH. Moreover, as δ ***decreases*** the action, so does ∂. Thus, we reduce the Morse-Bott situation to a Morse situation.

To start, we note that the maps in (44) are explicitly given by

$$\begin{aligned}
\iota\big[s\big]^{b-1} &= \big[s\big]^b && \text{for} && s \in FC^{b-1} \\
\pi\big[s\big]^b &= \big[n_b\big]_b^b && \text{for} && s = \textstyle\sum_{a\leq b} n_a \in FC^b \\
\delta\big[n_b\big]_b^b &= \big[dn_b\big]^{b-1} && \text{for} && n_b \in FC_b^b.
\end{aligned}$$

Let us make the following definition: $\aleph^b := \ker(\delta) = im(\pi) = \pi(FH^b) \subset FH_b^b$.

In the following, we are interested in bounded infinite sums $\sum_{b\leq b_0} \eta_b$ of classes $\eta^b \in \aleph^b$, i.e. in elements of the space

$$\aleph := \varinjlim_b \varprojlim_a \bigoplus_{a\leq c\leq b} \aleph^c = \left\{ \sum_{b\leq b_0} \eta^b \;\middle|\; b_0 \in \mathbb{Z}, \eta^b \in \aleph^b \right\}.$$

We want to define a surjective map $\Psi : \aleph \to FH$, which is R-linear if R is a field. For this purpose choose for every $\eta \in \aleph^b$ a preimage under π, i.e. fix a map

$$\rho : \aleph^b \to FH^b \qquad \text{such that} \qquad \pi(\rho(\eta)) = \eta \quad \text{and} \quad \rho(0) = 0. \tag{45}$$

Note that in general ρ is neither unique nor does there exist any R-linear ρ **!** However, one can pick an R-linear ρ if the sequence splits at π, in particular if R is a field or semi-simple. Now we define Ψ by setting

$$\Psi\left(\sum_{b\leq b_0} \eta^b\right) := \left[\sum_{b\leq b_0} s^b\right], \qquad \text{where} \quad [s^b]^b = \rho(\eta^b),\ s^b \in FC^b. \tag{46}$$

It is easy to see that Ψ is well-defined, as $\sum_{b\leq b_0} s^b$ is a cycle, whose class does not depend on the particular choice of the representative s^b for $\rho(\eta^b)$. Indeed we have:

$$d\Bigl(\sum_{b\le b_0} s^b\Bigr) = \sum_{b\le b_0} ds^b = \sum_{b\le b_0} 0 = 0 \qquad (s^b \in \ker(d) \;\; \forall b \le b_0)$$

$$\sum_{b\le b_0}(s^b + d\beta^b) = \sum_{b\le b_0} s^b + d\sum_{b\le b_0} \beta^b. \qquad (\beta^b \in FC^b)$$

As $\rho(0) = 0$, Ψ is also not effected by leading zeros, i.e. $\Psi\bigl(\sum_{b\le b_0} \eta^b\bigr) = \Psi\bigl(\sum_{b\le b_0-1} \eta^b\bigr)$ if $\eta^{b_0} = 0$. Note that Ψ does depend on the choice of ρ and is therefore in general not an R-homomorphism! However, if ρ is R-linear, then Ψ becomes R-linear as well.

Theorem 82 (Representation Theorem).
The homology FH is represented by $\aleph$ that is $\Psi(\aleph) = FH$, i.e. Ψ is surjective.

Remark. It follows from this theorem that FH as a set cannot be larger then $\aleph$. Note that if we have a grading of the complex, then this theorem and all following extend to the graded homology groups and reads as $\Psi(\aleph_k) = FH_k$.

Proof:
To show that Ψ is surjective let $\sigma \in FH$ be arbitrary with $[s^{b_0}] = \sigma$, $s^{b_0} \in FC$, such that the highest non-zero coefficient of s^{b_0} has index b_0. So $s^{b_0} \in FC^{b_0}$ and the class $\sigma^{b_0} := [s^{b_0}]^{b_0} \in FH^{b_0}$ is well-defined. We will construct for each $b \le b_0$ a quintuple

$$(\sigma^b, \eta^b, s^b, \mathfrak{s}^b, \beta^b) \in FH^b \times \aleph^b \times FC^b \times FC^b \times FC^b \qquad \text{where}$$
$$\eta^b = \pi(\sigma^b), \quad [s^b]^b = \sigma^b, \quad [\mathfrak{s}^b]^b = \rho(\eta^b) \quad \text{and} \quad \mathfrak{s}^b = s^b - s^{b-1} - d\beta^b. \qquad (*)$$

Then we see that $\Psi\bigl(\sum_{b\le b_0} \eta^b\bigr) = \sigma$, i.e. that Ψ is surjective, as

$$\Psi\Bigl(\sum_{b\le b_0} \eta^b\Bigr) = \Bigl[\sum_{b\le b_0} \mathfrak{s}^b\Bigr] = \Bigl[\sum_{b\le b_0} s^b - s^{b-1} - d\beta^b\Bigr]$$
$$= \Bigl[\sum_{b\le b_0}(s^b - s^{b-1}) - \sum_{b\le b_0} d\beta^b\Bigr] = \Bigl[s^{b_0} - d\sum_{b\le b_0} \beta^b\Bigr] = [s^{b_0}] = \sigma.$$

The first element $\sigma^{b_0} = [s^{b_0}]^{b_0}$ is already given. Now let σ^b, $b \le b_0$ be constructed and set η^b, s^b and $\mathfrak{s}^b$ as in $(*)$. Then we have

$$\pi(\sigma^b - \rho(\eta^b)) = \pi(\sigma^b) - \pi(\rho(\eta^b)) \overset{(45)}{=} \eta^b - \eta^b = 0.$$

Thus we have $\sigma^b - \rho(\eta^b) \in \ker(\pi) = im(\iota)$. Hence we can apply j and define $\sigma^{b-1} := j(\sigma^b - \rho(\eta^b)) \in FH^{b-1}$. Let s^{b-1} be a representative of σ^{b-1}. The definition of σ^{b-1} then yields

$$[s^{b-1}]^b = \iota\Bigl([s^{b-1}]^{b-1}\Bigr) = \iota(\sigma^{b-1}) \overset{(47)}{=} \sigma^b - \rho(\eta^b) = [s^b - \mathfrak{s}^b]^b.$$

Hence there exists a $\beta^b \in FC^b$ such that $s^{b-1} + d\beta^b = s^b - \mathfrak{s}^b$. This gives us the quintuple $(\sigma^b, \eta^b, s^b\mathfrak{s}^b, \beta^b)$ and a new σ^{b-1}. $\square$

In order to obtain more precise information, we are now going to analyze the "kernel" of Ψ, i.e. the preimage $\Psi^{-1}(0)$. To this purpose define

$$K^b := im(\delta) = \delta\big(FH_{b+1}^{b+1}\big) \cong FH_b^b \Big/ \pi(FH^b) \qquad \text{and} \qquad K := \varinjlim \varprojlim \bigoplus_{a\leq c\leq b} K^c,$$

where K is the space of bounded infinite sums $\sum_{k\leq b_0} \kappa^b$, $\kappa^b \in K^b$.
We want to define a map $\Phi : K \to \aleph$ with $\Phi(K) = \Psi^{-1}(0)$, such that Φ is R-linear if R is a field. Note that this implies that if Φ and Ψ are R-linear, then $FH \cong \aleph \Big/ \Phi(K)$. This will ensure that the homology of $\partial = \Phi \circ \delta$ is isomorphic to FH.
To define Φ, choose for every $\sigma \in im(\iota) \subset FH^b$ a preimage under ι, i.e. fix a map

$$j : im(\iota) \to FH^{b-1} \qquad \text{such that} \qquad \iota(j(\sigma)) = \sigma \quad \text{and} \quad j(0) = 0. \tag{47}$$

Again, j is in general only a map, but can be chosen R-linear if the sequence splits at ι. Then, for $\sum_{b\leq b_0} \kappa^b \in K$ set

$$\sigma^{b_0} := \kappa^{b_0} \in FH^{b_0}, \qquad \eta^{b_0} := \pi(\sigma^{b_0}) \in \aleph^{b_0}$$
$$\sigma^{b-1} := j\big(\sigma^b - \rho(\eta^b)\big) + \kappa^{b-1} \in FH^{b-1}, \qquad \eta^{b-1} := \pi(\sigma^{b-1}) \in \aleph^{b-1}.$$

Now, we define Φ by

$$\Phi\Big(\sum_{b\leq b_0} \kappa^b\Big) := \sum_{b\leq b_0} \eta^b \in \aleph.$$

As Φ does depend on ρ and j, it is in general not R-linear. However, if j and ρ are R-linear, the same holds for Φ. As $j(0) = 0$ and $\rho(0) = 0$, Φ is not effected by leading zeros, i.e. $\Phi\big(\sum_{b\leq b_0} \kappa^b\big) = \Phi\big(\sum_{b\leq b_0-1} \kappa^b\big)$ if $\kappa^{b_0} = 0$.

Lemma 83. *We have indeed $\Psi^{-1}(0) = \Phi(K)$.*

Proof:

- $\Psi^{-1}(0) \subset \Phi(K)$

  Proof: Let $\sum_{b\leq b_0} \eta^b \in \Psi^{-1}(0)$ be arbitrary with representatives $[s^b]^b = \rho(\eta^b)$ and define $\sigma^B := \big[\sum_{b\leq B} s^b\big]^B \in FH^B$. As $\Psi\big(\sum_{b\leq b_0} \eta^b\big) = \big[\sum_{b\leq b_0} s^b\big] = 0$, there exists a $b_0' \in \mathbb{Z}$ and $\beta \in FC^{b_0'}$ such that $\sum_{b\leq b_0} s^b = d\beta$. By setting $s^b = 0$ for $b \geq b_0$ we may without loss of generality assume that $b_0' = b_0 + 1$ (note that $\Psi\big(\sum_{b\leq b_0} \eta^b\big) = \Psi\big(\sum_{b\leq b_0-1} \eta^b\big)$ if $\eta^{b_0} = 0$).
  Now, we construct a sequence $\kappa^b \in K^b, b \leq b_0$ such that

  $$\kappa^{b_0} = \sigma^{b_0} \quad \text{and} \quad \sigma^{b-1} = j\big(\sigma^b - \rho(\eta^b)\big) + \kappa^{b-1}.$$

  By the construction of Φ, it then follows that $\Phi\big(\sum_{b\leq b_0} \kappa^b\big) = \sum_{b\leq b_0} \eta^b$.

  We saw above that $\sum_{b\leq b_0} s^b = d\beta$ with $\beta \in FC^{b_0+1}$. It follows that

  $$\iota(\sigma^{b_0}) = \iota\Big(\Big[\sum_{b\leq b_0} s^b\Big]^{b_0}\Big) = \Big[\sum_{b\leq b_0} s^b\Big]^{b_0+1} = 0$$

  and hence that $\sigma^{b_0} \in \ker(\iota) = im(\delta)$. The definition $\kappa^{b_0} := \sigma^{b_0} \in \delta\big(FH_{b_0+1}^{b_0+1}\big) = K^{b_0}$ is therefore correct.

For any $B \leq b_0$ we calculate

$$\iota\Big(\sigma^{B-1} - j\big(\sigma^B - \rho(\eta^B)\big)\Big) \overset{(47)}{=} \iota\Big(\big[\sum_{b\leq B-1} s^b\big]^{B-1}\Big) - \big(\sigma^B - \rho(\eta^B)\big)$$
$$= \big[\sum_{b\leq B-1} s^b\big]^B - \big[\sum_{b\leq B} s^b\big]^B + [s^B]^B \qquad = 0.$$

Therefore, we have $\sigma^{B-1} - j(\sigma^B - \rho(\eta^B)) \in \ker(\iota) = im(\delta)$ and hence there exists a $\kappa^{B-1} \in \delta(FH^{B-1}_{B-1})$ such that $\sigma^{B-1} = j\big(\sigma^B - \rho(\eta^B)\big) + \kappa^{B-1}$.

- $\Psi^{-1}(0) \supset \Phi(K)$

  Proof: Let $\sum_{b\leq b_0} \kappa^b \in K$ be arbitrary and let σ^b, η^b for $b \leq b_0$ be as in the definition of $\Phi\big(\sum_{b\leq b_0} \kappa^b\big)$. Let $[dn^{b+1}]^b = \kappa^b$, $[s^b]^b = \sigma^b$ and $[t^b]^b = \rho(\eta^b)$ be their representatives, where $s^b, t^b \in FC^b$ and $n^{b+1} \in FC^{b+1}_{b+1}$ for $b \leq b_0$. Then

  $$\Psi\Big(\Phi\Big(\sum_{b\leq b_0} \kappa^b\Big)\Big) = \Psi\Big(\sum_{b\leq b_0} \eta^b\Big) = \Big[\sum_{b\leq b_0} t^b\Big].$$

  It follows from the definition of Φ that $[s^{b_0}]^{b_0} = \sigma^{b_0} = \kappa^{b_0} = [dn^{b_0+1}]^{b_0}$. Hence we may without loss of generality assume that $s^{b_0} = dn^{b_0+1}$. It also follows from the definition of Φ that

  $$[s^{b-1}]^b = \iota(\sigma^{b-1}) = \sigma^b - \rho(\eta^b) + \iota(\kappa^{b-1}) = \big[s^b - t^b + dn^b\big]^b.$$

  Hence there exists a $\beta^b \in FC^b$ such that $s^{b-1} = s^b - t^b + dn^b + d\beta^b$. Then we have

  $$\sum_{b\leq b_0} t^b = \sum_{b\leq b_0} \big(s^b - s^{b-1} + dn^b + d\beta^b\big) = s^{b_0} + d\sum_{b\leq b_0}(n^b + \beta^b) = dn^{b_0+1} + d\sum_{b\leq b_0}(n^b + \beta^b).$$

  Thus, $\sum_{b\leq b_0} t^b$ is a boundary and hence $\Psi \circ \Phi\big(\sum_{b\leq b_0} \kappa^b\big) = \big[\sum_{b\leq b_0} t^b\big] = 0 \in FH$.

  $\square$

Lemma 84. $\Phi^{-1}(0) = \{0\}$, *i.e. if Φ is R-linear, then it is injective.*

Proof: Let $\sum_{b\leq b_0} \kappa^b \in \Phi^{-1}(0)$ be arbitrary. Let σ^b and η^b for $b \leq b_0$ be as in the definition of Φ. As $\Phi\big(\sum_{b\leq b_0} \kappa^b\big) = \sum_{b\leq b_0} \eta^b = 0$, we find that $\eta^b = 0$ for all $b \leq b_0$. Let $[dn^{b+1}]^b = \kappa^b$ and $[s^b]^b = \sigma^b$ be their representatives. Note that we may choose $[0]^b = \rho(\eta^b)$ as representatives, as $\eta^b = 0$ and $\rho(0) = 0$. The same calculations as in the second part of the proof of Lemma 83 now show with $t^b = 0$ that

$$0 = dn^{b_0+1} + d\textstyle\sum_{b\leq b_0}(n^b + \beta^b).$$

As $\sum_{b\leq b_0}(n^b + \beta^b) \in FC^{b_0}$, we find that $\kappa^{b_0} = [dn^{b_0+1}]^{b_0} = 0$. This implies that $0 = \Phi\big(\sum_{b\leq b_0} \kappa^b\big) = \Phi\big(\sum_{b\leq b_0-1} \kappa^b\big)$. Repeating the same arguments iteratively then shows that $\kappa^b = 0$ for all $b \leq b_0$ and therefore that $\sum_{b\leq b_0} \kappa^b = 0$. $\square$

We defined above $F\mathscr{C} := \varinjlim \varprojlim \bigoplus_{c=a}^{b} FH_c^c$. Note that $\aleph = \ker\delta$ is a natural subset of $F\mathscr{C}$. The limit of the maps $\delta : FH_{b+1}^{b+1} \to im(\delta) \subset FH^b$ is an R-linear map $\delta : F\mathscr{C} \to K$. If R is a field or semi-simple, we can choose Φ to be R-linear. As mentioned above, we then define an R-linear operator ∂ on $F\mathscr{C}$ by

$$\partial := \Phi \circ \delta.$$

Since $im(\Phi) \subset \aleph = \ker(\delta)$, we find that $\partial^2 = 0$, i.e. that ∂ is a boundary operator.

Theorem 85 (Reduction Theorem)**.**
Let R be a field (or semi-simple). Then, Ψ induces a filtration preserving isomorphism between the homology of $(F\mathscr{C}, \partial)$ and FH.
As $F\mathscr{C}$ is generated by FH_a^a, $a \in \mathbb{Z}$, we can hence in a Morse-Bott setup always algebraically pretend that FH is built from the singular homologies of the critical manifolds.

Proof: As R is a field (or semi-simple), we may choose Φ and Ψ to be R-linear. Then it follows from Lemma 84 that $\ker(\Phi) = \Phi^{-1}(0) = \{0\}$. This shows that Φ is injective and hence $\ker(\partial) = \ker(\Phi \circ \delta) = \ker(\delta) = \aleph$. Since $im(\partial) = \Phi(\delta(F\mathscr{C})) = \Phi(im(\delta)) = \Phi(K)$, we have for the homology of $(F\mathscr{C}, \partial)$ that

$$\ker(\partial) \Big/ im(\partial) = \aleph \Big/ \Phi(K) \, .$$

By Lemma 83, we have $\Phi(K) = \ker(\Psi)$. Hence, Ψ induces a well-defined injective map

$$\Psi : \ker(\partial) \Big/ im(\partial) = \aleph \Big/ \Phi(K) \longrightarrow FH, \tag{$*$}$$

which we denote by abuse of notation also Ψ. As Ψ is by Theorem 82 surjective, we find that $(*)$ is in fact an isomorphism. The action filtration preserving property is obvious by the construction of Ψ (see (46)). $\square$

Examples. Let us consider 4 critical points $C = \{a_1, b_1, a_0, b_0\}$, where the index denotes the action, i.e. $f(a_k) = k$. We set

$$m(b_1, a_1) = 2, \qquad m(b_0, a_1) = 1 \qquad \text{and} \qquad m(b_0, a_0) = 2.$$

$$\text{Then} \qquad da_1 = 2b_1 + b_0, \qquad da_0 = 2b_0 \qquad db_1 = db_0 = 0.$$

In a graded context, one should think of the a_k as having one index higher then the b_k. Using $\mathbb{Z}$-coefficients, we get

$$FH_1^1 = \langle b_1 \rangle \Big/ \langle 2b_1 \rangle \cong \mathbb{Z}_2 \qquad FH_0^0 = FH^0 = \langle b_0 \rangle \Big/ \langle 2b_0 \rangle \cong \mathbb{Z}_2$$

$$FH = FH_0^1 = \langle b_0, b_1 \rangle \Big/ \langle 2b_0, 2b_1 + b_0 \rangle = \langle 2b_1 + b_0, b_1 \rangle \Big/ \langle -4b_1, 2b_1 + b_0 \rangle \cong \mathbb{Z}_4.$$

This shows that there is no hope of building a chain complex from FH_1^1 and FH_0^0 whose homology is FH, if we do not use field coefficients.

If we take $\mathbb{Z}_2$-coefficients instead, we get $da_1 = b_0$ and $da_0 = db_1 = db_0 = 0$. Then

$$FH_1^1 = \langle a_1, b_1 \rangle \cong \left(\mathbb{Z}_2\right)^2, \qquad FH_0^0 = FH^0 = \langle a_0, b_0 \rangle \cong \left(\mathbb{Z}_2\right)^2,$$
$$FH = FH_0^1 = \langle a_0, b_0, b_1 \rangle \Big/ \langle b_0 \rangle \cong \left(\mathbb{Z}_2\right)^2.$$

The only relevant sequence here is

$$FH^0 \xrightarrow{\iota} FH^1 \xrightarrow{\pi} FH_1^1 \xrightarrow{\delta} FH^0,$$

where δ maps the class of b_1 to 0 and the class of a_1 to the class of b_0, as $da_1 = b_0$ and $db_1 = 0$. As $FH^0 = FH_0^0$ no further maps have to be constructed. The boundary operator ∂ on the complex $F\mathscr{C} = \big\langle [a_1], [a_0], [b_1], [b_0] \big\rangle$ is thus given by $\partial[a_1] = [b_0]$ and $\partial[a_0] = \partial[b_1] = \partial[b_0] = 0$. The homology of this complex is therefore

$$\big(F\mathscr{C}, \partial\big) = \big\langle [a_0], [b_0], [b_1] \big\rangle \Big/ \big\langle [b_0] \big\rangle \cong \left(\mathbb{Z}_2\right)^2,$$

which is trivially isomorphic to FH.

5. Contact surgery and handle attaching

This section is mostly already included in [12]. However, we will redo the line of arguments to fill in some delicate details left open in the original article. Thus, we hope to make the proof that symplectic (co)homology is invariant under subcritical surgery (Theorem 94) more transparent, at least to some readers.
First, we describe the general construction for contact surgery, which is done by attaching a symplectic handle H_k^{2n} to the symplectization of a contact manifold. Then, we describe the symplectic standard handle, which is a subset of $\mathbb{R}^{2n}$ defined as the intersection of two sublevel sets $\{\psi < -1\} \cap \{\phi > -1\}$, where ϕ and ψ are functions on $\mathbb{R}^{2n}$. While ϕ is explicitly given, we describe the construction of a suitable ψ in the Subsection 5.3. The calculation of Conley-Zehnder indices for 1-periodic Reeb orbits on H_k^{2n} concludes this section.

5.1. Surgery along isotropic spheres

Let us briefly recall the contact surgery construction due to Weinstein, [53]. Consider an isotropic sphere S^{k-1} in a contact manifold (N^{2n-1}, ξ). The 2-form $\omega = d\lambda$ for a contact form λ (with $\xi = \ker \lambda$) defines a natural conformal symplectic structure on ξ. Denote the ω-orthogonal on ξ by $\perp_\omega$. Since S is isotropic, it holds that $TS \subset TS^{\perp_\omega}$. So, the normal bundle of S in N is given by

$$TN/TS = TN/\xi \,\oplus\, \xi/(TS)^{\perp_\omega} \oplus (TS)^{\perp_\omega}/TS.$$

The Reeb field R_λ trivializes TN/ξ. The bundle $\xi/(TS)^{\perp_\omega}$ is canonically isomorphic to T^*S via $v \mapsto \iota_v\omega$. The conformal symplectic normal bundle $CSN(S) := (TS)^{\perp_\omega}/TS$ carries a natural conformal symplectic structure induced by ω. Since S is a sphere, the embedding $S^{k-1} \subset \mathbb{R}^k$ provides a natural trivialization of the bundle $\mathbb{R}R_\lambda \oplus T^*S$. This trivialization together with a conformally symplectic trivialization of $CNS(S)$ specifies a standard framing for S in N.
Note that we have to assume that $CNS(S)$ is trivializable. This holds certainly true for $S = S^0 = \{N, S\}$ (two points) or $S = S^{n-1}$. In the latter case we have $(TS)^{\perp_\omega} = TS$ and hence $CNS(S) = (0)$. Therefore, taking connected sums and surgery along Legendrian spheres is always possible.

Following Weinstein, we define an isotropic setup as a quintuple $(P, \omega, Y, \Sigma, S)$, where (P, ω) is a symplectic manifold, Y a Liouville vector field for ω, Σ a hypersurface transverse to Y (so Σ is contact) and S an isotropic submanifold of Σ. In [53], Weinstein proves the following variant of his famous neighborhood theorem for isotropic manifolds:

Proposition 86 (Weinstein). *Let $(P_0, \omega_0, Y_0, \Sigma_0, S_0)$ and $(P_1, \omega_1, Y_1, \Sigma_1, S_1)$ be two isotropic setups. Given a diffeomorphism from S_0 to S_1 covered by an isomorphism of their symplectic subnormal bundles, there exist neighborhoods U_j of S_j in P_j and an isomorphism of isotropic setups*

$$\phi : (U_0, \omega_0, Y_0, \Sigma_0 \cap U, S_0) \to (U_1, \omega_1, Y_1, \Sigma_1 \cap U_1, S_1)$$

which restricts to the given mappings on S_0.

We may now define contact surgery along an isotropic sphere as follows:
Let $H_k^{2n} \approx D^k \times D^{2n-k}$ be a symplectic standard handle (see 5.2) and let S^{k-1} be an isotropic sphere in a contact manifold (N^{2n-1}, ξ). Then, Proposition 86 allows us to glue the (lower) boundary $S^k \times D^{2n-k}$ of H_k^{2n} to the symplectization $N \times [0,1]$ along the boundary part $U_1 \cap N \times [0,1]$ of a tubular neighborhood U_1 of $S \times \{1\}$ (see Figure 2). We obtain an exact symplectic manifold $P := N \times [0,1] \cup_S H_k^{2n}$ with a Liouville vector field Y which is on $N \times [0,1]$ simply $\frac{\partial}{\partial t}$, where t denotes the coordinate on $[0,1]$. The field Y is inward pointing along $\partial^- P := N \times \{0\}$ and outward pointing along the other boundary component $\partial^+ P$. Both manifolds are hence contact and $\partial^+ P$ is obtained from N by surgery along S. Moreover, P is an exact symplectic cobordism between $\partial^- P$ and $\partial^+ P$.

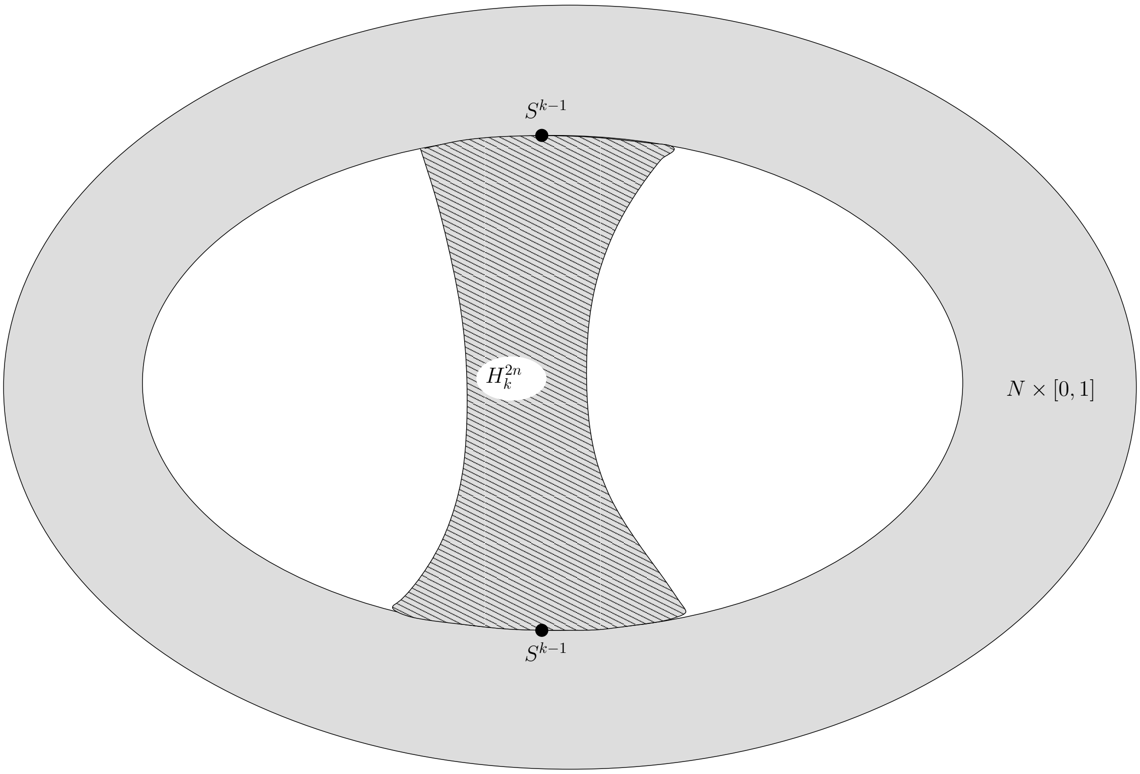

Fig. 2: $N \times [0,1]$ with handle attached

5.2. The handle H_k^{2n}

In order to specify a standard handle H_k^{2n}, we consider $\mathbb{R}^{2n}$ with symplectic coordinates $(q,p) = (q_1, p_1, \dots, q_n, p_n)$ and the following Weinstein structure (cf. [53]):

$$\begin{aligned}
\omega &:= \sum_{j=1}^{n} dq_j \wedge dp_j \\
Y &:= \sum_{j=1}^{k} \left(2q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right) + \sum_{j=k+1}^{n} \frac{1}{2} \left(q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j} \right) \\
\phi &:= \sum_{j=1}^{k} \left(q_j^2 - \frac{1}{2} p_j^2 \right) + \sum_{j=k+1}^{n} \frac{A_j}{4} \left(q_j^2 + p_j^2 \right),
\end{aligned}$$

with constants $A_j > 0$.
Observe that the Liouville vector field Y and the Weinstein function ϕ satisfy $Y \cdot \phi > 0$ away from the origin. Note that Y is in fact a Liouville vector field for ω, as $\mathcal{L}_X \omega = \omega$, and its associated Liouville 1-form $\lambda := \iota_X \omega$ satisfies $d\lambda = \omega$. Explicitly, λ is given by

$$\lambda := \sum_{j=1}^{k} (2q_j dp_j + p_j dq_j) + \sum_{j=k+1}^{n} \frac{1}{2} (q_j dp_j - p_j dq_j).$$

We introduce furthermore the following three quantities:

$$x := \sum_{j=1}^{k} q_j^2 \qquad y := \sum_{j=1}^{k} \frac{1}{2} p_j^2 \qquad z := \sum_{j=k+1}^{n} \frac{A_j}{4} \left(q_j^2 + p_j^2 \right),$$

whose Hamiltonian vector fields are given by

$$X_x = \sum_{j=1}^{k} 2q_j \frac{\partial}{\partial p_j} \quad X_y = \sum_{j=1}^{k} -p_j \frac{\partial}{\partial q_j} \quad X_z = \sum_{j=k+1}^{n} \frac{A_j}{2} \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

This convention allows us to write $\phi = x - y + z$ and $X_\phi = X_x - X_y + X_z$.
Now, consider the level surface $\Sigma^- := \{\phi = -1\}$ and note that Y is transverse to Σ^-, as $Y \cdot \phi|_{\Sigma^-} > 0$. Hence, $\lambda|_{T\Sigma^-}$ is a contact form. The set $S := \{x = z = 0,\ y = +1\}$ is an isotropic sphere in Σ^- and the quintuple $(\mathbb{R}^{2n}, \omega, Y, \Sigma^-, S)$ will be the isotropic setup where we glue H_k^{2n} to a contact manifold. To specify a handle H_k^{2n}, we choose a different Weinstein function $\psi(q,p) = \psi(x,y,z)$ on $\mathbb{R}^{2n}$ such that the following holds:

(ψ1) $\dfrac{\partial \psi}{\partial x}, \dfrac{\partial \psi}{\partial z} \geq 0, \quad \dfrac{\partial \psi}{\partial y} \leq 0, \quad$ and $\quad \dfrac{\partial \psi}{\partial x} = \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \psi}{\partial z} = 0 \quad$ only at the origin.

(ψ2) $\psi = \phi \quad$ for $\quad y > 1 + \varepsilon \quad$ with ε arbitrarily small.

(ψ3) The set $\{\psi < -1\} \cap \{\phi > -1\}$ is diffeomorphic to $D^k \times D^{2n-k}$.

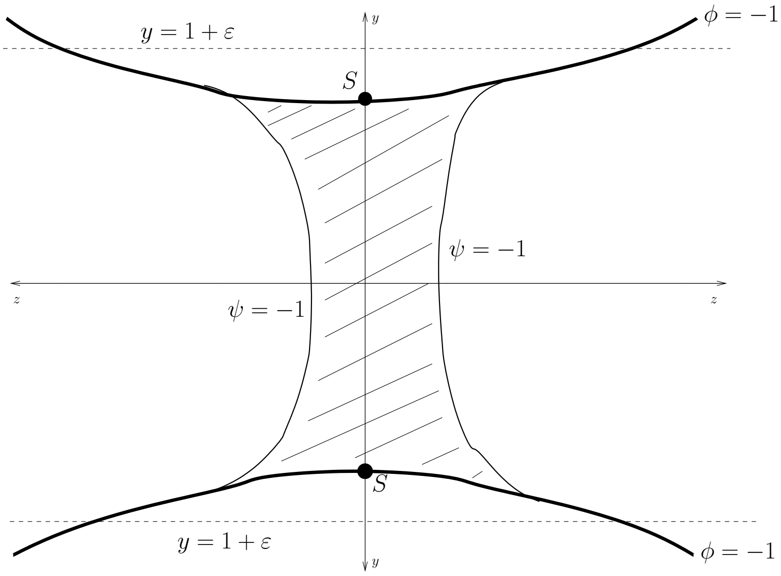

Fig. 3: The handle H_k^{2n}

The handle is defined to be the closure $\quad H_k^{2n} := \overline{\{\psi < -1\} \cap \{\phi > -1\}}$ (see Figure 3).

Properties.

- It follows from (ψ1) that the level sets $\Sigma^+ := \{\psi = -1\}$ and $\Sigma^- = \{\phi = -1\}$ are both contact hypersurfaces, as also $X \cdot \psi > 0$. They coincide for $y \geq 1 + \varepsilon$ due to (ψ2) and they contain the boundary of H_k^{2n}. Condition (ψ3) on the other hand assures that Σ^+ is obtained from Σ^- by surgery along S.
- It follows also from (ψ2) that reducing ε lets the handle become thinner. By choosing ε sufficiently small, we can make the handle so thin that its "lower" boundary $\{\phi = -1\} \cap H_k^{2n}$ lies inside any prescribed neighborhood of S.
- The handle stays unchanged if we take $\phi' = \alpha \cdot \phi + \beta$ and $\psi' = \alpha \cdot \psi + \beta$, $\alpha \neq 0$, provided that we set $\quad H_k^{2n} = \overline{\{\psi' < -\alpha + \beta\} \cap \{\phi' > -\alpha + \beta\}}$.

The Hamiltonian vector field $X_{\psi'}$ of $\psi' = \alpha \cdot \psi + \beta$ is given by

$$\begin{aligned} X_{\psi'} &= \alpha \cdot X_\psi = \alpha \cdot \left(\frac{\partial \psi}{\partial x} X_x + \frac{\partial \psi}{\partial y} X_y + \frac{\partial \psi}{\partial z} X_z \right) \\ &= \alpha \sum_{j=1}^{k} \left(2 \frac{\partial \psi}{\partial x} q_j \frac{\partial}{\partial p_j} - \frac{\partial \psi}{\partial y} p_j \frac{\partial}{\partial q_j} \right) + \alpha \sum_{j=k+1}^{n} \frac{\partial \psi}{\partial z} \cdot \frac{A_j}{2} \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right). \end{aligned} \tag{48}$$

For our symplectic setting, we consider the following Lyapunov function $f := \sum_{j=1}^{k} q_j p_j$. Note that f satisfies $X_{\psi'} \cdot f > 0$ away from the critical points of f. This shows that all periodic orbits of $X_{\psi'}$ are contained in the set

$$\{x = y = 0\} = \{q_1 = p_1 = \ldots = q_k = p_k = 0\}.$$

5.3. An explicit ψ and its extension to a neighborhood of H_k^{2n}

It is not difficult to find a Weinstein function ψ which satisfies (ψ1)–(ψ3). Fix $\varepsilon > 0$ and choose a smooth, monotone function $g : \mathbb{R} \to [0,1]$ such that

$$g(t) = \begin{cases} 0 & \text{for} \quad t \leq 0 \\ 1 & \text{for} \quad t \geq 1+\varepsilon \end{cases}$$

$$\text{and} \qquad 0 \leq g'(t) \leq \frac{1}{1+\varepsilon} + \delta, \qquad \delta > 0 \quad \text{small}, \quad \forall\, t.$$

$$\text{Then set} \qquad \psi := x - y + z - (1+\varepsilon/2) + (1+\varepsilon/2)\cdot g(y). \tag{49}$$

This satisfies (ψ1)–(ψ3), provided that δ is small enough.

For symplectic (co)homology, we need ψ not only on H_k^{2n}, but we have to arrange that ψ extends to a neighborhood of H_k^{2n} in a specific way. In order to describe what we mean by that, we make the following observation:
Let $\Sigma^- = \{\phi = -1\}$ and $\Sigma^+ = \{\psi = -1\}$ be as above. As both are hypersurfaces transversal to the Liouville vector field Y, the flow φ^t of Y provides symplectic embeddings of the symplectizations of Σ^+ resp. Σ^- into $\mathbb{R}^{2n}$:

$$\Phi^{\pm} : \Sigma^{\pm} \times (-\infty,\infty) \to \mathbb{R}^{2n}, \qquad \Phi^{\pm}(y,t) = \varphi^t(y)$$

On any symplectization $(\Sigma \times (-\infty,\infty), d(e^t\lambda))$ of a contact manifold and for any $\alpha, \beta \in \mathbb{R}$, we define a function h_Σ by $\qquad h_\Sigma(s,t) := \alpha \cdot e^t + \beta$.
We call such a function linear on $\Sigma \times \mathbb{R}$, and in fact it is linear when using the coordinate $r := e^t$, $r \in (0,\infty)$, instead of t. Observe that the Hamiltonian vector field of h_Σ is given by $X_{h_\Sigma}(s,t) = \alpha \cdot R_\lambda(s)$, where R_λ is the Reeb vector field of λ, the contact form on Σ. Let $\tilde{h}^{\pm}_{\Sigma}$ be functions of this form for $\Sigma^\pm$ with $\alpha = 1, \beta = -2$, such that $\tilde{h}_{\Sigma^\pm}(\Sigma^\pm) = -1$, and let $h^{\pm}_{\Sigma} := \tilde{h}_{\Sigma^\pm} \circ (\Phi^\pm)^{-1}$ be their pushforward onto the image of $\Phi^\pm$ in $\mathbb{R}^{2n}$. Note that h^+_Σ and h^-_Σ coincide on $\Phi^\pm((\Sigma^+ \cap \Sigma^-) \times (-\infty,\infty))$, as $\Phi^- = \Phi^+$ on $(\Sigma^- \cap \Sigma^+) \times (-\infty,\infty)$.
In order to compare the symplectic (co)homologies of Σ^- and Σ^+, we need a Hamiltonian that is linear on the negative symplectization of Σ^- and the positive symplectization of Σ^+. As ψ will serve as such a Hamiltonian, we require that $\psi = h^+_\Sigma$ on $\{\psi \geq -1\}$ and $\psi = h^-_\Sigma$ on $\{\phi \leq -1\} \setminus U$, where U is a compact neighborhood of $S = \{x = z = 0,\ y = 1\}$ (see Figure 4).

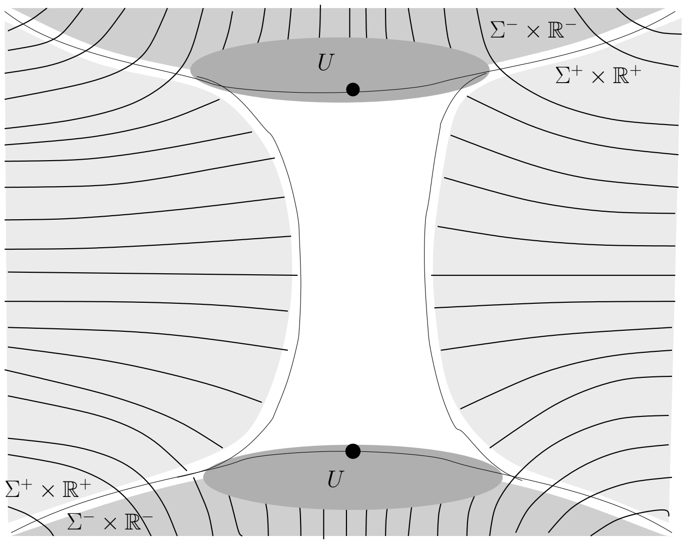

Fig. 4: Areas, where ψ is linear together with the symplectizations of $\Sigma^{\pm}$

Discussion 87. It is the extension of ψ beyond the handle, that is not quite correct in [12]: It is stated there that one can extend ψ on the positive symplectization of Σ^+ such that there is only one 1-periodic orbit of X_ψ on the handle. To achieve this, ψ has to be linear on the symplectizations away from the handle (as already stated above), i.e. ψ has to be of the form $\psi = \alpha \cdot e^t + \beta$ for $\alpha \notin spec(\Sigma^+)$. Moreover, it has to be of this form on the set $\{x = y = 0\}$ and it has to be increasing for $y \to 0$ on the set $\{x = z = 0\}$.
Let us write for the moment $\psi = \alpha^+ \cdot e^t + \beta^+$ on $\Sigma^+ \times \mathbb{R}^+$ and $\psi = \alpha^- \cdot e^t + \beta^-$ on $\Sigma^- \times \mathbb{R}^-$. As Σ^+ and Σ^- coincide together with their symplectizations on an open set, we find that $\alpha^+ = \alpha^-$ and $\beta^+ = \beta^-$. However, following the path depicted in Figure 5 and keeping in mind that $\psi = \alpha^+ \cdot e^t + \beta^+$ on $\{x = y = 0\}$ and $\partial_y \psi \leq 0$ on $\{x = z = 0\}$, we find that $\beta^+ > \beta^-$, a contradiction.

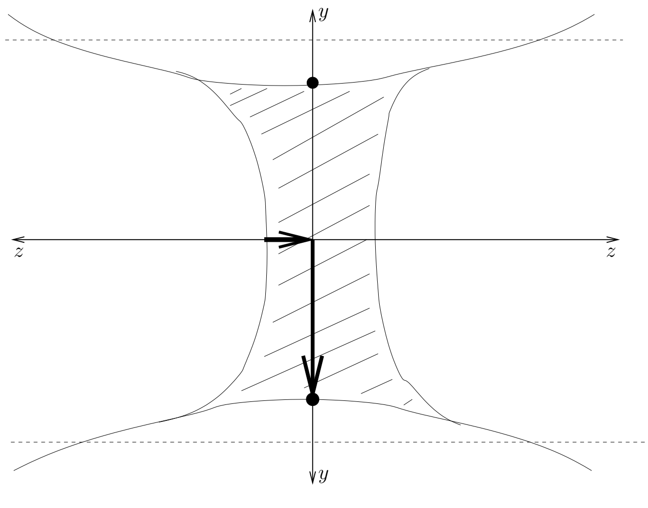

Fig. 5: Fig. 1: The problematic path

Our solution to this dilemma is to allow ψ to have a varying slope on $\{x = y = 0\}$, first letting it grow very slowly coming from the origin and increasing the slope sharply near Σ^+. Using the Lyapunov function f, we can then show that this construction creates 1-periodic X_ψ-orbits only in the set $\{x = y = 0\}$. These can be explicitly described and are hence still manageable.

In order to construct such a ψ, we need the following two technical lemma:

Lemma 88. *Consider $\mathbb{R}^{2n}$ with the standard symplectic structure and the Liouville vector field given in 5.2 and*

$$x := \sum_{j=1}^{k} q_j^2 \qquad y := \sum_{j=1}^{k} \tfrac{1}{2} p_j^2 \qquad z := \sum_{j=k+1}^{n} \tfrac{A_j}{4} \left(q_j^2 + p_j^2\right).$$

Let $\Sigma \subset \mathbb{R}^{2n}$ be a smooth hypersurface transverse to the Liouville vector field Y such that its outer normal N is of the form $N = c_x \cdot \nabla_x + c_y \cdot \nabla_y + c_z \cdot \nabla_z$, where $c_x, c_y, c_z : \Sigma \to \mathbb{R}$ are functions with $c_x, c_z > 0, c_y < 0$ and $\nabla_x, \nabla_y, \nabla_z$ are the gradients of x, y, z. Let $\tilde{h}_\Sigma$ denote the function $\tilde{h}_\Sigma(s,t) = \alpha e^t + \beta$ on the symplectization of Σ and let $h_\Sigma = \tilde{h}_\Sigma \circ \Phi^{-1}$ be its pushforward onto $\mathbb{R}^{2n}$ by the flow Φ of Y.
Then, the Hamiltonian vector field X_h of h_Σ is of the form

$$X_h = C_x \cdot X_x + C_y \cdot X_y + C_z \cdot X_z,$$

where $C_x, C_y, C_z \in C^\infty(\mathbb{R}^{2n})$ are functions satisfying $C_x, C_z > 0,\ C_y < 0$.

Remark.

- Note that $C_x, C_z > 0$ and $C_y < 0$ guarantees that the Lyapunov function $f = \sum_{j=1}^{k} q_j p_j$ satisfies $X_h \cdot f = 0$ only on $\{x = y = 0\}$.
- The assumptions on Σ are satisfied, if $\Sigma = \psi^{-1}(c)$ for a function ψ on x, y, z with $\frac{\partial \psi}{\partial x}\big|_\Sigma, \frac{\partial \psi}{\partial z}\big|_\Sigma > 0$ and $\frac{\partial \psi}{\partial y}\big|_\Sigma < 0$.

Proof: As $X_{\tilde{h}} = \alpha \cdot R_\lambda$ on $\Sigma \times (-\infty, \infty)$, it follows that on $\mathbb{R}^{2n}$ holds $X_h|_{\varphi^t(\Sigma)} = \alpha \cdot e^t \cdot R_t$, where R_t is the Reeb field of $\lambda_t := \lambda|_{T\varphi^t(\Sigma)}$, the contact form on $\varphi^t(\Sigma)$. By assumption, a normal N to Σ satisfies

$$\begin{aligned} N &= c_x \nabla_x + c_y \nabla_y + c_z \nabla_z \\ &= c_x \sum_{j=1}^{k} 2 q_j \frac{\partial}{\partial q_j} + c_y \sum_{j=1}^{k} p_j \frac{\partial}{\partial p_j} + c_z \sum_{j=k+1}^{n} \frac{1}{2} \left(q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j} \right). \end{aligned}$$

The flow of φ^t of Y is given by

$$\varphi^t = \Big(\underbrace{\dots, e^{2t} \cdot q_j, e^{-t} \cdot p_j, \dots}_{j=1,\dots,k}, \underbrace{\dots, e^{t/2} \cdot q_j, e^{t/2} \cdot p_j, \dots}_{j=k+1,\dots,n} \Big).$$

A normal N^t to $\varphi^t(\Sigma)$ is hence given by

$$N^t = e^{2t} \cdot c_x \nabla_x + e^{-t} \cdot c_y \nabla_y + e^{t/2} \cdot c_z \nabla_z.$$

Let J denote the standard almost complex structure on $\mathbb{R}^{2n}$, i.e. $J(\frac{\partial}{\partial q_j}) = \frac{\partial}{\partial p_j}$ and $J(\frac{\partial}{\partial p_j}) = -\frac{\partial}{\partial q_j}$. Using the definition of λ_t and N^t, we find that the Reeb vector field R_t is given by

$$R_t = \frac{JN^t}{\lambda(JN^t)} = \frac{1}{C}\left(e^{2t}c_x X_x + e^{-t}c_y X_y + e^{t/2}c_z X_z\right)$$
$$\text{with } C = \lambda(JN^t) = e^{4t}(c_x)\cdot 4x + e^{-2t}c_y \cdot (-2y) + e^t c_z \cdot z \; > 0.$$

Using $X_h|_{\varphi^t(\Sigma)} = \alpha \cdot e^t R_t$, we see that X_h is exactly of the announced form. □

Lemma 89. *Let $\varepsilon, \delta, c > 0$ be constants. Then there exists a smooth monotone function $g : \mathbb{R} \to [0,1]$ such that*

$$g(t) = 0 \quad \textit{for} \quad t \leq 0 \qquad g(t) = 1 \quad \textit{for} \quad t \geq \varepsilon \tag{$*$}$$

and for all C^1-functions ϕ, ψ with $\phi(0) = \psi(0)$ and $|\frac{\partial}{\partial t}\phi(t) - \frac{\partial}{\partial t}\psi(t)| < c$ for all $t \in [0,\varepsilon]$ holds that

$$\left|\left|\frac{\partial}{\partial t}\Big(\phi + (\psi - \phi)\cdot g\Big) - \Big(\frac{\partial}{\partial t}\phi + (\frac{\partial}{\partial t}\psi - \frac{\partial}{\partial t}\phi)\cdot g\Big)\right|\right|_\infty \leq \delta. \tag{$**$}$$

In other words, we can interpolate between ϕ and ψ, such that the slope of the interpolation is arbitrary close to the interpolation of the slopes of ϕ and ψ.

Proof: We calculate that

$$\frac{\partial}{\partial t}\Big(\phi + (\psi - \phi)g\Big) = \Big(\frac{\partial}{\partial t}\phi + (\frac{\partial}{\partial t}\psi - \frac{\partial}{\partial t}\phi)g\Big) + (\phi - \psi)\frac{\partial}{\partial t}g.$$

Therefore, $(**)$ translates to

$$\left|(\psi - \phi)\frac{\partial}{\partial t}g\right| = \left|\int_0^t \left(\frac{\partial}{\partial s}\psi - \frac{\partial}{\partial s}\phi\right) ds \cdot \frac{\partial}{\partial t}g\right| \leq c\cdot t \cdot \frac{\partial}{\partial t}g \leq \delta \qquad \forall t \in [0,\varepsilon].$$

So, $(**)$ is satisfied, if $0 \leq \frac{\partial}{\partial t}g(t) \leq \delta/ct \; \forall\, t \in [0,\varepsilon]$. As $\int_0^\varepsilon \delta/ct\, dt = \infty$, we can choose a smooth function $\tilde{g}$ satisfying

$$0 \leq \tilde{g}(t) \leq \frac{\delta}{c\cdot t}, \qquad \tilde{g} \equiv 0 \text{ for } t \leq 0 \text{ and } t \geq \varepsilon \qquad \text{and} \qquad \int_0^\varepsilon \tilde{g}(t)\, dt = 1.$$

Setting $g(t) := \int_0^t \tilde{g}(s)\, ds$ then gives the desired function. □

Now, we construct ψ in two steps:

- First, recall that the isotropic sphere $S \subset \Sigma^- = \{\phi = -1\}$ is given by

  $$S := \{x = z = 0,\ y = 1\}.$$

  Consider the function h_Σ^-. As the Reeb vector field R_{Σ^-} of $(\Sigma^-, \lambda|_{T\Sigma^-})$ coincides with the Hamiltonian vector field X_ϕ, we find $X_{h_\Sigma^-} = R_{\Sigma^-} = X_\phi$ and $dh_\Sigma^- = d\phi$.

As also $h_\Sigma^-(\Sigma^-) = \phi(\Sigma^-) = -1$, we find that h_Σ^- and ϕ coincide up to first order on Σ^-. Therefore, given any neighborhood U of S, there exists a function $\hat\phi$ of x, y, z and a neighborhood $\hat U \subset U$, such that $\hat\phi \equiv h_\Sigma^-$ on $\mathbb{R}^{2n} \setminus U$, $\hat\phi \equiv \phi$ on $\hat U$ and $\hat\phi$ is arbitrarily C^1-close to h_Σ^-. Consequently, we can arrange that

$$X_{\hat\phi} = C_x \cdot X_x + C_y \cdot X_y + C_z \cdot X_z \quad \text{with} \quad C_x, C_z > 0,\ C_y < 0.$$

Now choose a handle H_k^{2n} so thin, such that $H_k^{2n} \cap \Sigma^- \subset \hat U$. See Figure 6 for the different areas. Let H_k^{2n} be defined by a function $\tilde\psi$ as in (49) and set

$$\hat\psi : \{\phi \le -1\} \cup H_k^{2n} \to \mathbb{R} \qquad \hat\psi = \begin{cases} \tilde\psi & \text{on } \big(\hat U \cap \{\phi \le -1\}\big) \cup H_k^{2n} \\ \hat\phi & \text{on } \big(U \cap \{\phi \le -1\}\big) \setminus \hat U \\ h_\Sigma^- & \text{on } \{\phi \le -1\} \setminus U \end{cases} .$$

Since $\tilde\psi = \hat\phi$ outside a small neighborhood of H_k^{2n}, we find that $\hat\psi$ is smooth.

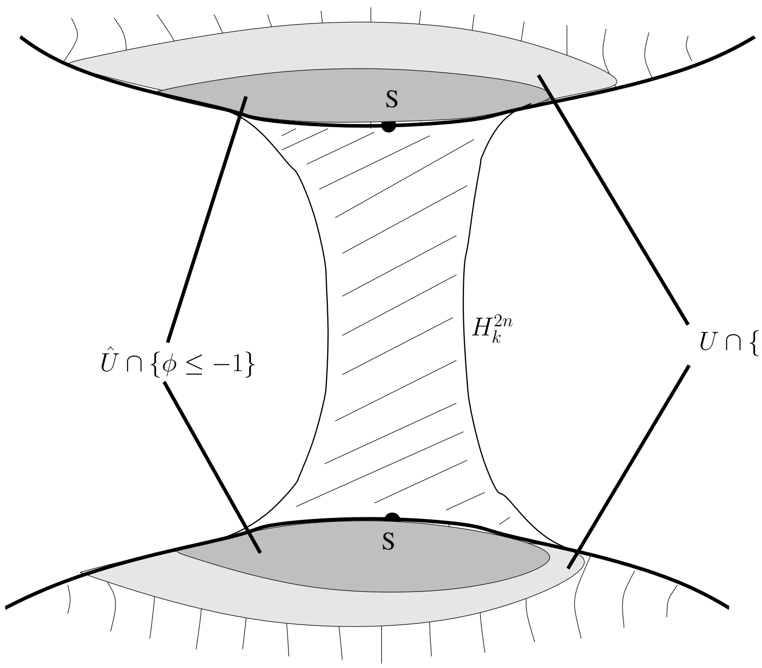

Fig. 6: Areas, where $\hat\psi$ is defined

- For the second step consider the function h_Σ^+ associated to $\Sigma^+ = \{\hat\psi = -1\}$. Use Lemma 89 to construct for $\delta > 0$ small a smooth monotone function g with

$$g(t) = 0 \quad \text{for} \quad t \le -1-\delta, \qquad g(t) = 1 \quad \text{for} \quad t \ge -1$$

such that $\quad \psi := \hat\psi + (h_\Sigma^+ - \hat\psi) \cdot g(h_\Sigma^+)$ satisfies

$$X_\psi = C_x \cdot X_x + C_y \cdot X_y + C_z \cdot X_z, \qquad C_x, C_z > 0,\ C_y < 0. \quad (50)$$

Recall that $h_\Sigma^+ = h_\Sigma^- = \hat\psi$ outside U, so that we are actually interpolating along a compact set. Moreover, it follows from this, that $\psi = h_\Sigma^+$ away from the handle.

5.4. Closed orbits and Conley-Zehnder indices

Note that (50) implies for the Lyapunov function f that $f \cdot X_\psi = 0$ only on $\{x = y = 0\}$. This continues to hold, when we consider $\psi' := \alpha \cdot \psi + \beta$ and it guarantees that the only periodic orbits of $X_{\psi'}$ lie in $\{x = y = 0\}$. In the following we will determine the 1-periodic orbits of $X_{\psi'}$ and calculate their Conley-Zehnder indices. By (50), the Hamiltonian vector field $X_{\psi'}$ is given by

$$X_{\psi'} = \alpha \cdot \big(C_x X_x + C_y X_y + C_z X_z\big),$$

where $C_x, C_y, C_z \in C^\infty(\mathbb{R}^{2n})$ are functions with $C_x, C_z > 0$, $C_y < 0$ and X_x, X_y and X_z are the Hamiltonian vector fields of x, y, z and given by

$$X_x = \sum_{j=1}^{k} 2q_j \frac{\partial}{\partial p_j}, \qquad X_y = \sum_{j=1}^{k} -p_j \frac{\partial}{\partial q_j}, \qquad X_z = \sum_{j=k+1}^{n} \frac{A_j}{2}\left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j}\right).$$

Note that on $\{x = y = 0\}$, we have therefore

$$X_{\psi'} = \alpha \cdot C_z \sum_{j=k+1}^{n} \frac{A_j}{2}\left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j}\right).$$

On $\Sigma^+ \times \mathbb{R}^+ \cap \{x = y = 0\}$, we have by our construction $\psi' = \alpha \cdot h_\Sigma^+$ and hence that $X_{\psi'} = \alpha \cdot R_\lambda|_{\Sigma^+}$, where the Reeb vector field on $\Sigma^+ \cap \{x = y = 0\}$ is given by

$$R_\lambda|_{\Sigma^+} = \frac{2}{\varepsilon} \cdot \sum_{j=k+1}^{n} \frac{A_j}{2}\left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j}\right) = \frac{2}{\varepsilon} X_z.$$

The constant $2/\varepsilon$ comes from the fact that $\lambda(X_z) = z$ and that the value of z on $\Sigma^+ \cap \{x = y = 0\} = \psi^{-1}(-1) \cap \{x = y = 0\}$ is given by (49) as

$$-1 = 0 - 0 + z - (1 + \varepsilon/2) \qquad \Leftrightarrow \qquad z = \varepsilon/2.$$

On the other hand, for $z < \varepsilon/2 - O(\delta)$, we have by the second step that $\psi = \hat{\psi}$. By (49) we have for $y = 0$ that $\hat{\psi} = x - y + z - (1 + \varepsilon/2)$ and hence there $X_{\psi'} = \alpha X_z$. It follows that on $\{x = y = 0\}$, we have $X_{\psi'} = \alpha \cdot C_z X_z$, where C_z is a z-dependent interpolation between the constants 1 and $2/\varepsilon$. Now we can calculate the flow φ^t of $X_{\psi'}$ on $\{x = y = 0\}$. First we calculate for $z(\varphi^t(p, q))$

$$\frac{d}{dt} z\big(\varphi^t\big) = dz(X_{\psi'}) = \alpha \cdot C_z\, dz(X_z) = 0.$$

It follows that z is constant along the flow lines of φ. Now, consider for $j = k+1, \dots, n$ the complex coordinates $z_j = q_j + i p_j$. Then, we have on $\{x = y = 0\}$:

$$X_{\psi'} = \alpha C_z \cdot \Bigg(\underbrace{0, \dots, 0}_{j=1,\dots,k}, \underbrace{\dots, iA_j \cdot z_j, \dots}_{j=k+1,\dots,n}\Bigg).$$

As $z(\varphi^t)$ is independent from t and hence $\frac{d}{dt}C_z(0,0,z(\varphi^t)) = 0$, we obtain that the flow φ^t of $X_{\psi'}$ on $\{x = y = 0\}$ is given by

$$\varphi^t(0, z_{k+1}, \dots, z_n) = \Big(0, \dots, 0, \exp\big(i\alpha C_z A_{k+1} t\big) \cdot z_{k+1}, \dots, \exp\big(i\alpha C_z A_n t\big) \cdot z_n\Big). \qquad (51)$$

By choosing the constants A_j linear independent over $\mathbb{Q}$, we can arrange that the 1-periodic orbits of $X_{\psi'}$ on $\{x = y = 0\}$ are isolated. They are given by values of z and j such that

$$\alpha A_j \cdot C_z(0,0,z) \in 2\pi\mathbb{Z},$$

except for the one constant orbit at the origin. For α appropriately chosen, we can assume that there are only finitely many such orbits.
Now, we calculate the Conley-Zehnder indices of these orbits, using the definition of μ_{CZ} for a path of symplectic matrices from Section 3.3. Let $z^0 \in \{x = y = 0\}$ be such that $\gamma(t) := \varphi^t(z^0), 0 \le t \le 1$, $\gamma(0) = \gamma(1)$, is a 1-periodic orbit of $X_{\psi'}$. In order to calculate $\mu_{CZ}(\gamma)$, we identify $T_{\gamma(t)}\mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$ in the natural way. This yields a path Φ_ψ in $Sp(2n)$ given by $\Phi_\psi(t) = D\varphi^t(z^0)$. The t-derivative of Φ_ψ on $\{x = y = 0\}$ is given by

$$\begin{aligned} \frac{d}{dt}\Phi_\psi(t) &= \frac{d}{dt}D\varphi^t(z^0) = D\left(\frac{d}{dt}\varphi^t(z^0)\right) = DX_{\psi'}\big(\varphi^t(z^0)\big) \\ &= D\big(C_x X_x + C_y X_y + C_z X_z\big)\big(\varphi^t(z^0)\big) \\ &= \alpha \cdot diag\Bigg(\underbrace{\dots, \begin{pmatrix} 0 & -C_y \\ 2C_x & 0 \end{pmatrix}, \dots}_{j=1,\dots,k}, \underbrace{\dots, iC_z A_j, \dots}_{j=k+1,\dots,n}\Bigg) \circ \Phi_\psi(t). \end{aligned}$$

Note that no derivatives of C_x or C_y are involved, as $X_x = X_y = 0$ on $\{x = y = 0\}$. It follows that Φ_ψ is of block form $\Phi_\psi = diag\big(\Phi_\psi^1, \dots, \Phi_\psi^n\big)$, where the paths of 2×2 matrices Φ_ψ^j are solutions of an ordinary differential equation with initial value $\Phi_\psi^j(0) = \mathbb{1}$ and

$$\begin{aligned} &\tfrac{d}{dt}\Phi_\psi^j(t) = \alpha\begin{pmatrix} 0 & -C_y \\ 2C_x & 0 \end{pmatrix}\Phi_\psi^j(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2\alpha C_x & 0 \\ 0 & \alpha C_y \end{pmatrix}\Phi_\psi^j(t) \quad && j = 1, \dots, k \\ &\tfrac{d}{dt}\Phi_\psi^j(t) = i\alpha A_j C_z \cdot \Phi_\psi^j(t) && j = k+1, \dots, n. \end{aligned}$$

As the matrix $\left(\begin{smallmatrix} 2\alpha C_x & 0 \\ 0 & \alpha C_y \end{smallmatrix}\right)$ has for all t one positive and one negative eigenvalue, it follows that its signature is always zero and hence by the Robbin-Salamon definition of μ_{CZ} that $\mu_{CZ}(\Phi_\psi^j) = 0,\ j = 1, \dots, k$. For $j = k+1, \dots, n$ on the other hand, we find by Lemma 59 that

$$\mu_{CZ}\big(\Phi_\psi^j\big) = \left\lfloor\frac{\alpha A_j C_z}{2\pi}\right\rfloor + \left\lceil\frac{\alpha A_j C_z}{2\pi}\right\rceil, \qquad j = k+1, \dots, n$$

$$\text{and therefore}\quad \mu_{CZ}(\gamma) = \mu_{CZ}(\Phi_\psi) = \sum_{j=k+1}^{n}\left(\left\lfloor\frac{\alpha A_j C_z}{2\pi}\right\rfloor + \left\lceil\frac{\alpha A_j C_z}{2\pi}\right\rceil\right). \qquad (52)$$

Note again that $1 \le C_z \le 2/\varepsilon$ is constant for each γ with $\alpha A_j C_z(0,0,z(\gamma(0)) \in 2\pi\mathbb{Z}$. We have therefore $\mu_{CZ}(\gamma) \ge \alpha \cdot \sum\lceil A_j/2\pi\rceil$. It follows for $\alpha \to \infty$ that all possible values for $\mu_{CZ}(\gamma)$ go to ∞.

6. Symplectic (co)homology

In this section, we roughly introduce symplectic (co)homology – a Floer type homology similar to Rabinowitz-Floer homology. In fact, we will see in 6.6 that both constructions are closely related.
In Rabinowitz-Floer homology, the chain complex is built solely on data coming from the contact hypersurface Σ, a fact that makes this homology very suitable for calculating contact invariants. The chain complex in symplectic (co)homology uses all 1-periodic orbits of a Hamiltonian vector field X_H on V. Moreover, the definition of the homology involves direct limits indexed over the set of all admissible H. This makes the construction more flexible and allows us to prove the invariance of symplectic (co)homology under subcritical handle attachment, which we then transfer to Rabinowitz-Floer homology.

6.1. Setup

First, we describe shortly the setup for symplectic (co)homology. It is quite similar to the one for Rabinowitz-Floer homology, but has some important differences – in particular the use of time-dependent Hamiltonians and the absence of the parameter η.

In this section, let (V, λ) be a compact Liouville domain, with exact symplectic form $\omega = d\lambda$ and convex contact boundary $(\Sigma = \partial V, \alpha = \lambda|_{T\Sigma})$. As before, let Y be the Liouville vector field of λ. That (Σ, α) is convex simply means that Y points out of V along $\partial V = \Sigma$. The completion of V is still denoted $\widehat{V}$, as well as R denotes the Reeb vector field of α. The action spectrum of (Σ, α) contains in this section only *positive* periods of closed Reeb orbits, so that

$$spec(\Sigma, \alpha) := spec(\Sigma, \alpha) \cap \mathbb{R}^+.$$

Hamiltonians are smooth S^1-families of functions $H_t : \widehat{V} \to \mathbb{R}$ with Hamiltonian vector fields X_H^t given by

$$\omega(\cdot, X_H^t) = dH_t. \qquad \text{(for } t \in S^1 \text{ fixed)}$$

The action of a loop $x : S^1 \to \widehat{V}$ is defined by

$$\mathcal{A}^H(x) = \int_0^1 x^*\lambda - \int_0^1 H_t(x(t))dt.$$

The critical points of this functional are exactly the one-periodic solutions of

$$\dot{x}(t) = X_H^t. \tag{53}$$

We denote the set of these solutions by $\mathcal{P}(H)$.
Almost complex structures J_t come in S^1-families. An $\mathcal{A}^H$-gradient trajectory $u : \mathbb{R} \times S^1 \to \widehat{V}$ is in this section a solution of the Floer equation:

$$\partial_s u - \nabla \mathcal{A}^H = 0 \qquad \Leftrightarrow \qquad u_s + J(u_t - X_H^t) \ = 0. \tag{54}$$

As before, we are also interested in homotopies H_s of Hamiltonians. In this setting an $\mathcal{A}^{H_s}$-gradient trajectory is again a solution of (54), but with X_H^t depending on s.

Symplectic (co)homology can also be $\mathbb{Z}$-graded by the Conley-Zehnder index. For that, we require the same assumptions as for Rabinowitz-Floer homology, namely that $\Sigma = \partial V$ is simply connected and that the evaluation of the first Chern class $c_1(TV)$ vanishes on $\pi_2(V)$. One could also define the index under more relaxed conditions. However, imposing these assumptions makes it easier to compare SH and RFH.

Remark. Note that $\mu_{CZ}(x)$ of a 1-periodic orbit x of X_H^t is for SH calculated with respect to a trivialization of TV over a capping disc $\bar{u}$ of x and not with respect to a trivialization of the contact form ξ as in RFH. Nevertheless, our assumptions imply for a non-constant Reeb orbit x that $\mu_{CZ}(x)$ is the same for TV and ξ. This is due to the fact that over Σ, we have the splitting

$$TV = \xi \oplus \mathbb{R}R \oplus \mathbb{R}Y.$$

As the action of the linearized flow of X_H on $\mathbb{R}R \oplus \mathbb{R}Y$ is trivial (as both vector fields are preserved by X_H), we find by the product property of μ_{CZ} that $\mu_{CZ}(x)$ actually involves only the ξ part of TV. This phenomenon is illustrated by the calculations of μ_{CZ} on Brieskorn manifolds Σ_a in Section 7.1, where by Corollary 98 Y_2 is the Reeb vector field on Σ_a and X_2 the Liouville vector field on the filling W_ε.

For the construction of symplectic (co)homology we look at solutions of (54) such that $\lim\limits_{s \to \pm\infty} = x_\pm(t)$ are 1-periodic solutions of (53). In general, these solutions might not stay in a compact subset of $\widehat{V}$, even for $x_\pm$ fixed. Hence, it could be that the moduli space of these solutions is not compactifiable. To avoid this problem, we make the following restrictions:

- We call a Hamiltonian H *admissible*, writing $H \in Ad(V, \Sigma)$, if all its 1-periodic orbits are non-degenerate and if H is linear at infinity, that is if there exists an $R \in \mathbb{R}$ such that on $\Sigma \times [R, \infty) \subset \widehat{V}$ the Hamiltonian is of the form

$$H = h(e^r) = \alpha \cdot e^r + \beta$$

  with $\alpha, \beta \in \mathbb{R}, \alpha > 0$ and $\alpha \notin Spec(\Sigma, \lambda)$.

- We call a homotopy H_s between admissible Hamiltonians $H_\pm$ admissible if there exists an $R \in \mathbb{R}$ such that on $\Sigma \times [R, \infty)$ the homotopy has the form

$$H_s = h_s(e^r) \qquad \text{with} \qquad \partial_s \partial_r H_s \leq 0 \qquad \text{on} \qquad \Sigma \times [R, \infty).$$

- We call a Hamiltonian/homotopy H *weakly admissible*, writing $H \in Ad^w(V, \Sigma)$, if there exists an R such that on $\Sigma \times [R, \infty)$ it has the form

$$H = h\big(e^{r - f(y)}\big) = \alpha \cdot e^{r - f(y)} + \beta \quad \text{resp.} \quad H_s = h_s(e^{r - f_s(y)})$$

  for a function $f : \Sigma \to \mathbb{R}$. In the homotopy case we require that

$$(\partial_s \partial_r h_s)\big(e^{r - f_s(y)}\big) - (\partial_r h_s)\big(e^{r - f_s(y)}\big) \cdot \partial_s f_s(y) \leq 0, \quad \text{with } < 0 \text{ on } supp_s\, \partial_s f.$$

  If $\partial_r^2 h = 0$ (e.g. if h is linear), then this is equivalent to $\partial_s \partial_r H_s \leq 0$.

- We call a possibly s-dependent almost complex structure J (weakly) admissible, if it is cylindrical and time independent at infinity, that is if

$$d\big(e^{r-f_s}\big)\circ J_s = -\lambda \qquad \text{on } \Sigma\times[R,\infty)$$

for an $R\in\mathbb{R}$. We may write this shorter as $d(e^{r_s})\circ J=\lambda$ for $r_s := r-f_s$.

Note that in Rabinowitz-Floer homology, we required that H is constant outside a compact set. This implied by a Lemma of McDuff, [35] Lem. 2.4, that all solutions of the Rabinowitz-Floer equation (3) stayed in this compact set. The following lemma is a generalization of this crucial fact to Hamiltonians that are linear at infinity.

Lemma 90 (**Maximum Principle**).
Let H be a (weakly) admissible Hamiltonian/homotopy and J an admissible almost complex structure. Let $x_\pm\in P(H_\pm)$, where $H_\pm$ are the ends of the possibly constant homotopy H_s. Then there exists a constant $\sigma\le 1$ such that for $H_{\sigma\cdot s}$ and $J_{\sigma\cdot s}$ any solution $\boldsymbol{u}$ of (54) with $\lim\limits_{s\to\pm\infty} u(s)=x_\pm$ satisfies

$$e^r\circ u(s,t)\le e^C \qquad \forall (s,t)\in\mathbb{R}\times S^1$$

for some constant $C\ge R$ not depending on u. If H is a (weak) Hamiltonian or a non-weak homotopy, then we may choose $\sigma=1$, i.e. the Maximum principle holds already for H and J.

Proof: Our proof is a generalization of similar proofs by A.Oancea,[41], and P.Seidel, [48]. We give the proof only for homotopies H_s, which includes the Hamiltonian case by constant $H_s=H$. Let us consider the function $\rho:\mathbb{R}\times S^1\to\mathbb{R}$ given by

$$\rho := e^{r-f_s}\circ u = e^{r_s}\circ u, \qquad \text{where } r_s := r-f_s.$$

To ease the notation, we will drop the index s, writing only H,h,f and J. Moreover, we write h' instead of $\partial_r h$. However, we keep r_s and we write u_s,u_t for $\partial_s u$ and $\partial_t u$.

<u>Calculation of $\Delta\rho$</u>

$$\begin{aligned}\partial_s\rho = d\big(e^{r_s}\big)(u_s)+\big(\partial_s e^{r-f_s}\big)(u) &= d\big(e^{r_s}\big)\big(-J(u_t-X_H)\big)+e^{r-f_s}(u)\cdot(-\partial_s f)(u)\\ &= \lambda(u_t)+\lambda(X_H)-\rho\cdot(\partial_s f)(u)\\ &= \lambda(u_t)-\rho\cdot h'(\rho)-\rho\cdot(\partial_s f)(u),\end{aligned}$$

as $\lambda(X_H)=\omega(Y,X_H)=dH(\partial_r)=\partial_r H=\rho\cdot h'(\rho)$. Moreover, we have

$$\partial_t\rho = d\big(e^{r_s}\big)(u_t)=d\big(e^{r_s}\big)(Ju_s-X_H)=-\lambda(u_s),$$

as the orbits of X_{H_s} stay in the level sets of e^{r_s} and hence $d\big(e^{r_s}\big)(X_{H_s})=0$.

Therefore, we obtain for the Lapacian of ρ

$$\begin{aligned}
\Delta\rho &= \partial_s\Big(\lambda(u_t) - \rho\cdot\big[h'(\rho) + (\partial_s f)(u)\big]\Big) - \partial_t\lambda(u_s) \\
&= d\lambda(u_s,u_t) - \lambda(\underbrace{[u_s,u_t]}_{=0}) - \partial_s\rho\cdot(\partial_s f)(u) - \Big(-\rho(\partial_s f)(u) + \underbrace{d\big(e^{r_s}\big)(u_s)\Big)\cdot h'(\rho)}_{=dH(u_s)=d\lambda(u_s,X_H)} \\
&\quad - \rho\cdot\Big[(\partial_s h')(\rho) + h''(\rho)\cdot\partial_s\rho + (\partial_s^2 f)(u) + d(\partial_s f)(u_s)\Big] \\
&= \omega(u_s,\underbrace{u_t - X_H}_{=Ju_s}) - \partial_s\rho\cdot\big((\partial_s f)(u) + \rho\cdot h''(\rho)\big) - \rho\cdot d(\partial_s f)(u_s) \\
&\quad - \rho\cdot\Big[(\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u) + (\partial_s^2 f)(u)\Big] \\
&= |u_s|^2 - \partial_s\rho\cdot\big((\partial_s f)(u) + \rho\cdot h''(\rho)\big) - \rho\cdot d(\partial_s f)(u_s) \\
&\quad - \rho\cdot\Big[(\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u) + (\partial_s^2 f)(u)\Big].
\end{aligned}$$

Abbreviating $g(u) := (\partial_s f)(u) + \rho\cdot h''(\rho)$, we find that this is equivalent to

$$\Delta\rho + \partial_s\rho\cdot g(u) = |u_s|^2 - \rho\cdot d(\partial_s f)(u_s) - \rho\cdot\Big[(\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u) + (\partial_s^2 f)(u)\Big]. \quad (*)$$

Now if for $C > R$ holds on $[C,\infty)\times\Sigma$ that the right-hand side of $(*)$ is non-negative, then ρ satisfies on $[C,\infty)\times\Sigma$ a maximum principle and cannot have a local maximum at an interior point of $u^{-1}\big([C,\infty)\times\Sigma\big)$. As the asymptotics of u lie outside of $[C,\infty)\times\Sigma$, it follows that $\rho = e^{r-f_s}\circ u \le e^C$ everywhere.

Estimate of $\kappa := |u_s|^2 - \rho\cdot d(\partial_s f)(u_s)$

At first glance, this term might be an unbounded from below. However, as the Liouville form $\lambda = e^r\cdot\lambda_0$ grows exponentially in r, we will see that κ is in fact bounded by a constant, independent of u. Indeed, as $d(\partial_s f)$ is an r-invariant 1-form, there exists a vector field ξ_s on Σ, such that

$$d(\partial_s f)(\cdot) = d\lambda(\tfrac{1}{e^r}\xi_s,\cdot) \;\Rightarrow\; \rho\cdot d(\partial_s f)(u_s) = d\lambda(\xi_s,u_s).$$

For $c := \sup_s|J\xi_s|$, we find that this last expression is bounded by $c\cdot|u_s|$. It will be usefull to introduce σ at this point. Note that if we replace f_s by $f_{\sigma\cdot s}$, then κ becomes $|u_s|^2 - \sigma\cdot\rho\cdot d(\partial_s f)(u_s)$. Then, we have

$$\kappa = |u_s|^2 - \sigma\cdot\rho\cdot d(\partial_s f)(u_s) \ge |u_s|^2 - \sigma\cdot c\cdot|u_s| \ge -\tfrac{1}{4}c^2\cdot\sigma^2. \quad (**)$$

Here, the last estimate is the minimum of the parabola $x^2 - c\sigma x$. Finally note that outside the s-support of $\partial_s f$, we have $\kappa = |u_s|^2 \ge 0$.

Estimate of the whole right-hand side of $(*)$

Let us introduce σ everywhere in $(*)$. Then, we get the following

$$\begin{aligned}
\Delta\rho + \partial_s\rho\; g(u) \;&=\; |u_s|^2 - \sigma\rho\; d(\partial_s f)(u_s) - \rho\Big[\sigma(\partial_s h')(\rho) - \sigma\; h'(\rho)(\partial_s f)(u) + \sigma^2(\partial_s^2 f)(u)\Big] \\
&\overset{(**)}{\ge} -\rho\Big[\sigma\big((\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u)\big) + \sigma^2(\partial_s^2 f)(u)\Big] - \tfrac{1}{4}c^2\cdot\sigma^2. \quad (***)
\end{aligned}$$

For weakly admissable, we assumed that $(\partial_s h') - h'(\rho)(\partial_s f)(u) \leq 0$ with < 0 on the s-support of $\partial_s f$. As this support is bounded, we find for σ sufficiently small that the expression in the brackets is non-positive. Fixing such a σ, we find that for $\rho > R$ sufficiently large that the right-hand side is in fact non-negative. This proves the lemma. □

Remark.

- By decreasing σ, we can in fact achieve that $C = R$.
- If H is a Hamiltonian or a non-weak homotopy, then the term $(\partial_s^2 f)(u)$ does not exist and there is no need for a reparametrization by σ, i.e. we can choose $\sigma = 1$.

6.2. Symplectic homology

For a (weakly) admissible Hamiltonian H, we define the Floer homology $FH_*(H)$ as follows: The chain groups $FC_*(H)$ are the $\mathbb{Z}_2$-vector space generated by $\mathcal{P}(H)$. Note that due to $h' \notin Spec(\Sigma, \alpha)$ and the non-degeneracy of the 1-periodic orbits, we find that $\mathcal{P}(H)$ is in fact a finite set. Thus, $FC_*(H)$ is a finite vector space of dimension $|\mathcal{P}(H)|$. For $x_\pm \in \mathcal{P}(H)$ let $\widehat{\mathcal{M}}(x_-, x_+)$ denote the space of solutions u of (54) with $\lim\limits_{s \to \pm\infty} u = x_\pm$. There is an $\mathbb{R}$-action on this space given by time shift. The quotient under this action is called the moduli space of $\mathcal{A}^H$-gradient trajectories between x_- and x_+ and denoted by $\mathcal{M}(x_-, x_+) := \widehat{\mathcal{M}}(x_-, x_+)/\mathbb{R}$.
For a generic J, the space $\mathcal{M}(x_-, x_+)$ is a manifold. Its zero-dimensional component $\mathcal{M}^0(x_-, x_+)$ is compact and hence a finite set. Let $\#_2\mathcal{M}^0(x_-, x_+)$ denote its cardinality modulo 2. We define the operator $\partial : FC_*(H) \to FC_*(H)$ as the linear extension of

$$\partial x := \sum_{y \in \mathcal{P}(H)} \#_2\mathcal{M}^0(y, x) \cdot y.$$

A standard argument in Floer theory, involving the compactification of $\mathcal{M}^1(y, x)$, shows that $\partial^2 = 0$, so that ∂ is a boundary operator. We set as usual

$$FH_*(H) := \frac{\ker \partial}{\operatorname{im} \partial}.$$

To a (weakly) admissible homotopy H_s between admissible Hamiltonians $H_\pm$ we consider for $x_\pm \in \mathcal{P}(H_\pm)$ the moduli space of s-dependent $\mathcal{A}^{H_s}$-gradient trajectories $\mathcal{M}_s(x_-, x_+)$. Note that we have no time shift on this space, as equation (54) now depends on s. We define the continuation map $\sigma_*(H_-, H_+) : FC_*(H_+) \to FC_*(H_-)$ as the linear extension of

$$\sigma_*(H_-, H_+)x_+ = \sum_{x_- \in \mathcal{P}(H_-)} \#_2\mathcal{M}_s^0(x_-, x_+) \cdot x_-.$$

By considering homotopies of homotopies, one sees that $\sigma_*(H_-, H_+)$ is independent of the chosen homotopy. By considering the compactification of $\#_2\mathcal{M}_s^1(x_-, x_+)$, we obtain from Floer theory that $\partial \circ \sigma_* = \sigma_* \circ \partial$, so that $\sigma_*(H_-, H_+)$ is a chain map, which descends to a map $\sigma_*(H_-, H_+) : FH(H_+) \to FH(H_-)$.

For three admissible Hamiltonians H_1, H_2 and H_3, we have the composition rule

$$\sigma_*(H_1, H_3) = \sigma_*(H_1, H_2) \circ \sigma_*(H_2, H_3).$$

Observe that admissibility of a homotopy H_s between H_- and H_+ implies that $H_- > H_+$ on $\Sigma \times [R, \infty)$ for R sufficiently large. We introduce a partial ordering $\prec$ on $Ad^w(V, \Sigma)$ by saying $H_+ \prec H_-$ if and only if $H_+ < H_-$ on $\Sigma \times [R, \infty)$ for a sufficiently large $R \in \mathbb{R}$. This ordering together with the maps $\sigma_*(H_-, H_+)$ turn $(FH(H), \sigma)$ into a direct system over the directed set $(Ad^w(V, \Sigma), \prec)$. The symplectic homology groups $SH_*(V)$ are then defined to be the direct limit of this system:

$$SH_*(V) := \varinjlim FH_*(H).$$

A cofinal sequence $(H_n) \subset Ad^w(V, \Sigma)$ is a sequence of Hamiltonians such that $H_n \prec H_{n+1}$ and for any $H \in Ad^w(V, \Sigma)$ there exists an $n \in \mathbb{N}$ such that $H \prec H_n$. It follows from Theorem 74 that we have for any cofinal sequence

$$SH_*(V) = \lim_{n \to \infty} FH_*(H_n).$$

Finally, for any cofinal sequence there exist sequences $(R_n), (\alpha_n), (\beta_n) \subset \mathbb{R}$ and $(f_n) \subset C^\infty(\Sigma)$ such that (α_n) and (R_n) are monotone increasing and

$$H_n = \alpha_n \cdot e^{r - f_n} + \beta_n \qquad \text{on } \Sigma \times [R_n, \infty).$$

6.3. Truncation

For a (weakly) admissible Hamiltonian H and $b \in \mathbb{R}$ consider the subchain groups

$$FC_*^{<b}(H) \subset FC_*(H)$$

which are generated by whose $x \in \mathcal{P}(H)$ with $\mathcal{A}^H(x) < b$. For $a < b$, we set

$$FC_*^{[a,b)}(H) := FC_*^{<b}(H) \Big/ FC_*^{<a}(H)$$

We call $FC_*^{[a,b)}(H)$ truncated chain groups in the action window $[a, b)$. By setting $a = -\infty$, they include the cases $FC_*^{[-\infty,b)}(H) = FC_*^{<b}(H)$. Analogously one defines

$$FC_*^{\leq b}(B),\ FC_*^{>b}(H) := FC_*(H) \Big/ FC_*^{\leq b}(H),\ \ FC_*^{\geq b}(H),$$
$$FC_*^{(a,b]}(H),\ FC_*^{(a,b)}(H) \text{ and } FC_*^{[a,b]}(H).$$

Note that $FC_*^{[a,b)}(H) = FC_*^{(a,b)}(H)$ if $a \notin \mathcal{A}^H(\mathcal{P}(H))$. In the following, we restrict ourself for simplicity to $FC_*^{(a,b)}(H)$. However, most of the following results hold also for all other versions of action windows.

The following Lemma 91 shows that the boundary operator ∂ reduces the action. It induces therefore a boundary operator ∂ on the truncated chain groups and for this ∂ we define

$$FH_*^{(a,b)}(H) := \frac{\ker \partial}{\operatorname{im} \partial}.$$

Lemma 91. *If H is a Hamiltonian or a monotone decreasing homotopy and u a solution of (54) with $\lim\limits_{s\to\pm\infty} u = x_\pm \in \mathcal{P}(H)$, then $\mathcal{A}^H(x_+) \geq \mathcal{A}^H(x_-)$.*

Proof:

$$\begin{aligned}\mathcal{A}^H(x_+) - \mathcal{A}^H(x_-) &= \int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{A}^H(u(s))ds \\ &= \int_{-\infty}^{\infty} ||\nabla \mathcal{A}^H||^2 ds - \int_{-\infty}^{\infty}\int_0^1 \left(\frac{d}{ds}H\right)(u(s))dt\,ds \geq 0.\end{aligned}$$

Note that the second term is zero, if H does not depend on s, i.e. if H is a Hamiltonian. This shows that the monotone decreasing condition is only needed for homotopies. $\square$

Let H_-, H_+ be two (weakly) admissible Hamiltonians such that $H_- > H_+$ everywhere. Then we may choose a monotone decreasing (weakly) admissible homotopy H_s between them and it follows from Lemma 91 that the associated continuation map $\sigma_*(H_-, H_+)$ also decreases action. We obtain hence a well-defined map

$$\sigma_*(H_-, H_+) : FH_*^{(a,b)}(H_+) \to FH_*^{(a,b)}(H_-).$$

The truncated symplectic homology in the action window (a, b) is then defined as the direct limit under these maps:

$$SH_*^{(a,b)}(V) := \varinjlim FH_*^{(a,b)}(H).$$

<u>Attention</u>: Without further restrictions, we have always

$$SH^{(a,b)}(V) = 0, \qquad\qquad \text{whenever} \quad a > -\infty.$$

To see this, take any cofinal sequence of Hamiltonians (H_n) and take an increasing sequence $(\beta_n) \subset \mathbb{R}$ such that $\beta_n > \max\limits_{x\in\mathcal{P}(H_n)} \mathcal{A}^{H_n}(x)$. Then we find that $K_n := H_n + \beta_n - a$ yields also a cofinal sequence, but now

$$\max_{x\in\mathcal{P}(K_n)} \mathcal{A}^{K_n}(x) = \max_{x\in\mathcal{P}(H_n)} \mathcal{A}^{H_n}(x) - \beta_n + a < a$$

such that $FC_*^{(a,b)}(K_n) = FH_*^{(a,b)}(K_n) = 0$ for all n and hence $SH^{(a,b)}(V) = 0$. One can overcome this obstacle by restricting further the set of admissible Hamiltonians. For us, it will be enough to require that all Hamiltonians H are smaller then 0 on a fixed contact hypersurface $M \subset V$. We write $SH^{(a,b)}(V, M)$ for the direct limit of these Hamiltonians. In addition, we remark that for the definition of $FH_*^{(a,b)}(H)$ it suffices that only the 1-periodic orbits x of X_H with $\mathcal{A}^H(x) \in (a, b)$ are non-degenerate, as the others are discarded. Therefore, we call a Hamiltonian H admissible for $SH_*^{(a,b)}(V, M)$, writing $H \in Ad^{(a,b)}(V, M)$, if it satisfies

- $H|_M < 0$
- $H|_{\Sigma\times[R,\infty)} = h(e^r)$ for R large
- all $x \in \mathcal{P}^{(a,b)}(H) = \{x \in \mathcal{P}(H) \mid \mathcal{A}^H(x) \in (a, b)\}$ are non-degenerate.

The partial ordering on $Ad^{(a,b)}(V,M)$ is given by $H \prec K$ if $H < K$ everywhere. Similar, one defines weakly admissible Hamiltonians. Note that we are free to choose for the computation of $SH^{(a,b)}(V,M)$ cofinal sequences (H_n) which are also admissible for the whole symplectic homology or cofinal sequences, where the 1-periodic orbits of X_{H_n} are only non-degenerate in the action window (a,b).
When taking a cofinal sequence $(H_n) \subset Ad(V)$, we find that the projection

$$FC_*(H) \to FC_*^{>b}(H) = FC_*(H) \Big/ FC_*^{\leq b}(H)$$

or the short exact sequence

$$0 \to FC_*^{(a,b)}(H) \to FC_*^{(a,c)}(H) \to FC_*^{(b,c)}(H) \to 0$$

induce in homology the map

$$FH_*(H) \to FH_*^{\geq b}(H)$$

respectively the long exact sequence

$$\cdots \to FH_*^{(a,b)}(H) \to FH_*^{(a,c)}(H) \to FH_*^{(b,c)}(H) \to \dots$$

Applying the direct limit then yields the map

$$SH_*(V) \to SH_*^{>b}(V,M)$$

and (as $\varinjlim$ is an exact functor) the long exact sequence

$$\cdots \to SH_*^{(a,b)}(V,M) \to SH_*^{(a,c)}(V,M) \to SH_*^{(b,c)}(V,M) \to \dots$$

6.4. Symplectic cohomology

By dualizing the constructions from the previous section, we obtain the symplectic cohomology. Explicitly, we define for a (weakly) admissible Hamiltonian H the cochain groups $FC^*(H)$ again as the $\mathbb{Z}_2$-vector space generated by $\mathcal{P}(H)$. The coboundary operator δ is then defined as the linear extension of

$$\delta x := \sum_{y \in \mathcal{P}(H)} \#_2 \mathcal{M}^0(x,y) \cdot y.$$

Note that the operator δ increases action. The analogue construction of chain maps $\sigma^*(H_-,H_+)$ associated to an admissible homotopy H_s between Hamiltonians H_- and H_+ yields hence a map in the opposite direction (compared to $\sigma_*(H_-,H_+)$)

$$\sigma^*(H_-,H_+) : FH^*(H_-) \to FH^*(H_+),$$

where $H_- > H_+$ on $\Sigma \times [R,\infty)$ for R sufficiently large. It obeys the composition rule

$$\sigma^*(H_1,H_3) = \sigma^*(H_2,H_3) \circ \sigma^*(H_1,H_2).$$

By taking the same partial ordering on $Ad^w(V)$ as for homology, we obtain hence an inverse system. The symplectic cohomology $SH^*(V)$ is then defined to be the inverse limit of this system

$$SH^*(V) := \varprojlim FH^*(H).$$

Again, it can be calculated using cofinal sequences (H_n) of admissible Hamiltonians.
For the truncated version of symplectic cohomology, we now have to consider

$$FC^*_{>a}(H) \subset FC^*(H)$$

generated by those 1-periodic orbits with action greater then a. Then, we define

$$FC^*_{(a,b]}(H) := FC^*_{>a}(H) \Big/ FC^*_{>b}(H)$$

and all other truncated groups accordingly. As δ increases action, it is well-defined on the truncated chain groups and yields analogously $FH^*_{>a}(H)$ and $FH^*_{(a,b)}(H)$ as cohomology groups. Then considering only (globally) monotone decreasing homotopies, the chain maps σ^* are also well-defined on truncated groups and we obtain as inverse limits

$$SH^*_{>a}(V,M) = \varprojlim FH^*_{>a}(H), \qquad SH^*_{(a,b)}(V,M) = \varprojlim FH^*_{(a,b)}(H),$$

where we restricted again to $H \in Ad^w(V,M)$.
Unlike to the homology case, the long exact sequence

$$\cdots \to FH^*_{(b,c)}(H) \to FH^*_{(a,c)}(H) \to FH^*_{(a,b)}(H) \to \ldots$$

induces in general not a long exact sequence in symplectic cohomology. This is due to the fact that, in general, the inverse limit is not an exact functor, but only left exact (see Section 4 or [4] resp. [21]). However, the inclusion $FC^*_{>a}(H) \to FC^*(H)$ still induces a map

$$SH^*_{>a}(V,M) \to SH^*(V).$$

6.5. Transfer morphism and handle attaching

In the following, we construct a map $\pi_*(W,V) : SH_*(V) \to SH_*(W)$ for an exactly embedded Liouville subdomain $W \subset V$, as first suggested by Viterbo in [52]. Analogously, we construct a map $\pi^*(W,V) : SH^*(W) \to SH^*(V)$ in cohomology. Then we will show that $\pi_*(W,V)$ and $\pi^*(W,V)$ are isomorphisms if V is obtained from W by attaching a subcritical handle H^{2n}_k as described in Section 5.
As shown above, we have always maps $SH_*(V) \to SH^{>0}_*(V,\partial W)$ and $SH^*_{>0}(V,\partial W) \to SH^*(V)$. The maps $\pi_*(W,V)$ and $\pi^*(W,V)$ are obtained by showing the identities $SH^{>0}_*(V,\partial W) = SH_*(W)$ and $SH^*_{>0}(V,\partial W) = SH^*(W)$. This is done by giving an explicit cofinal sequence $(H_n) \subset Ad(V,\partial W)$.
The following proposition is based on ideas by Viterbo, [52]. Its proof is taken from McLean, [37]. We include it here for completeness and to add a missing argument for the homotopy case. See also Cieliebak, [12], for a slightly different approach.

Proposition 92. *There exists a cofinal sequence $(H_n) \subset Ad(V, \partial W)$ and a sequence of monotone decreasing admissible homotopies $(H_{n,n+1})$ between them such that*

1. $K_n := H_n|_W$, $K_{n,n+1} := H_{n,n+1}|_W$ *are sequences of admissible Hamiltonians / homotopies on* (W, ω).
2. *all 1-periodic orbits of* X_{H_n} *in* W *have positive action and all 1-periodic orbits of* X_{H_n} *in* $V \setminus W$ *have negative action.*
3. *all* $\mathcal{A}^H$*-gradient trajectories of* H_n *or* $H_{n,n+1}$ *connecting 1-periodic orbits in* W *are entirely contained in* W.

Proof: It will be convenient to use $z = e^r$ rather than r for the second coordinate in the completion $(\widehat{W}, \widehat{\omega}) = \big(W \cup (\partial W \times [1, \infty),\, d(z\alpha)\big)$. Note that we can embed $\widehat{W}$ into $\widehat{V}$ using the flow of Y_λ, where $\lambda = z\alpha$. The cylindrical end $\partial W \times [1, \infty)$ is then a subset of $\widehat{V}$. The first coordinates will be denoted z_W for $\partial W \times (0, \infty)$ and z_V for $\partial V \times (0, \infty)$.

To begin, assume that $Spec(\partial W, \lambda)$ and $Spec(\partial V, \lambda)$ are discrete and let

$$k : \mathbb{N} \to \mathbb{R} \setminus \Big(Spec(\partial W, \lambda) \cup 4 \cdot Spec(\partial V, \lambda)\Big)$$

be an increasing function such that $k(n) \to \infty$. Let $\mu : \mathbb{N} \to \mathbb{R}$ be defined by

$$\mu(n) = dist\big(k(n), Spec(\partial W, \lambda)\big) = \min_{a \in Spec(\partial W, \lambda)} |k(n) - a|.$$

Choose an increasing sequence $A = A(n)$ with

$$A > \frac{2k}{\mu} > 1 \qquad \text{and} \qquad A(n+1) > 2A(n)$$

which satisfies additionally the conditions $(\oplus)$ and $(\oplus\oplus)$ below. Note that we can always achieve $\frac{2k}{\mu} > 1$, as we may choose k arbitrarily large whilst making μ arbitrarily small. Let also $\varepsilon(n) > 0$ be a sequence tending to zero.
We assume that $H_n|_W$ is a C^2-small negative Morse function inside $W \setminus \big(\partial W \times [1-\varepsilon, 1)\big)$ and for $1 - \varepsilon \le z_W \le A$ of the form $H_n = g(z)$ with $g(1) = -\varepsilon$, $g' \ge 0$ and $g' \equiv k(n)$ for $1 \le z_W \le A - \varepsilon$. For $A \le z_W \le 2A$ we assume that $H_n \equiv B$ is constant with B being arbitrarily close to $k \cdot (A - 1)$.
Now we describe H_n on $\partial V \times [1, \infty)$: We keep H_n constant until we reach $z_V = 2A + P$, where P is some constant such that $\{z_W \le 1\} \subset \{z_V \le P\}$, which implies $\{z_W \le 2A\} \subset \{z_V \le 2A + P\}$. Then let $H_n = f(z_V)$ for $z_V \ge 2A + P$ with $0 \le f' \le \frac{1}{4}k(n)$ and $f' \equiv \frac{1}{4}k(n)$ for $z_V \ge 2A + P + \varepsilon$. Figure 7 gives a schematic illustration of H_n.

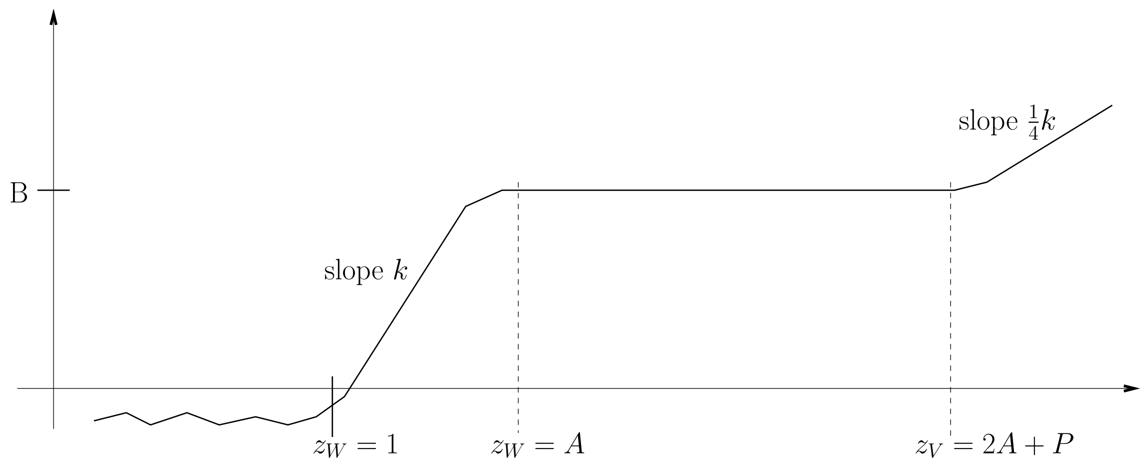

Fig. 7: The Hamiltonian H_n

As the action of an X_H-orbit on level $z = const.$ is $h'(z) \cdot z - h(z)$, we distinguish five types of 1-periodic orbits of X_H:

- critical points inside W of action > 0 (as H is negative and C^2-small inside W)
- non-constant orbits near $z_W = 1$ of action $\approx 1 \cdot g'(z) > 0$
- non-constant orbits on $z_W = a$ for a near A of action $\approx g'(a) \cdot a - B < (k - \mu) \cdot A - B \approx -\mu \cdot A + k < -k \to -\infty$
- critical points in $A < z_W$; $z_V < 2A + P$ of action $-B < 0$
- non-constant orbits on $z_V = a$ for a near $2A + P$ of action $\approx f'(a) \cdot a - B \leq \frac{1}{4}k \cdot (2A + P + \varepsilon) - B \approx -\frac{1}{2}kA + k \cdot (\frac{1}{4}P + \frac{1}{4}\varepsilon + 1) < 0$ for A sufficiently large (this is condition $(\oplus)$).

Obviously, (H_n) satisfies 1. and 2. of the proposition's claims. It only remains to show that $\mathcal{A}^H$-gradient trajectories connecting two orbits of non-negative action are contained entirely inside $z_W \leq 1$. By Gromov's Monotonicity Lemma (see [49], Prop. 4.3.1 and [42], Lem. 1) there exists a $K > 0$ such that any J-holomorphic curve which intersects $z_W = A$ and $z_W = 2A$ has area greater than KA. Note that inside $A \leq z_W \leq 2A$ the equation (54) reduces to an ordinary J-holomorphic curve equation, as $X_H \equiv 0$ there. Any $\mathcal{A}^H$-gradient trajectory connecting two orbits of non-negative action which intersects $z_W = A$ and $z_W = 2A$ has therefore area greater than KA – in other words the action difference between its ends is greater than KA.
For $k(n)$ fixed, the maximal action difference of two 1-periodic orbits in W is bounded from above. So for $A(n)$ sufficiently large (this is condition $(\oplus\oplus)$) no such $\mathcal{A}^H$-gradient trajectory can touch $z_W = 2A$. It follows then from the Maximum Principle that in fact all these $\mathcal{A}^H$-gradient trajectories have to remain inside $z_W \leq 1$.
For the construction of the homotopies $H_{n,n+1}$ we have to sharpen this argument. As $A(n+1) > 2A(n)$, we can take for $H_{n,n+1}$ the following interpolations:

At first, in time $s \in [0, 1/2]$, decrease H_{n+1} in the area $z_W \leq 2A(n)$ to H_n and keep it unchanged in $z_W \geq A(n+1)$. Then decrease in time $s \in [1/2, 1]$ the remaining part to H_n (see Figures 8 and 9).
For $s \in [-\infty, 1/2]$ the homotopy $H_{n,n+1}$ is then constant $B(n+1)$ in the area $A(n+1) \leq z_W \leq 2A(n+1)$ so that no $\mathcal{A}^H$-gradient trajectory can leave $z_W \leq 1$ in this time interval.
For $s \in [1/2, \infty]$ the homotopy $H_{n,n+1}$ is constant $B(n)$ in the area $A(n) \leq z_W \leq 2A(n)$ so that again no $\mathcal{A}^H$-gradient trajectory can leave $z_W \leq 1$ in this time interval. $\square$

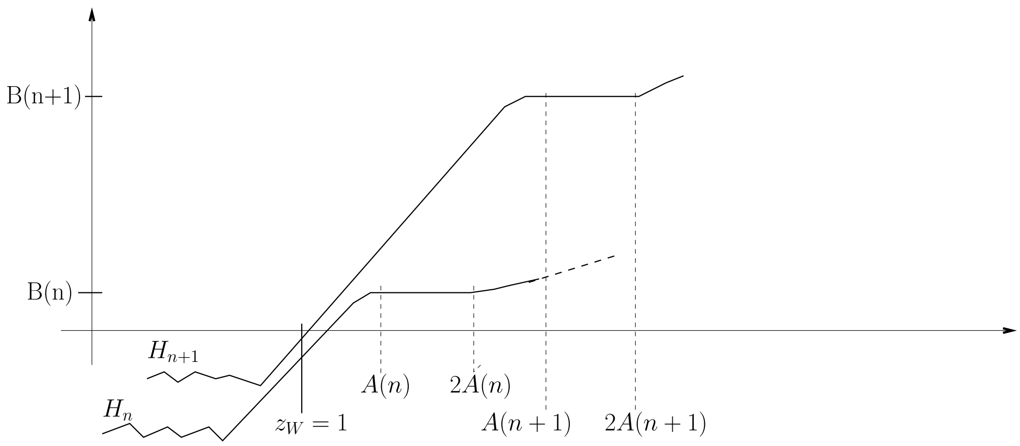

Fig. 8: Two Hamiltonian H_n and H_{n+1}

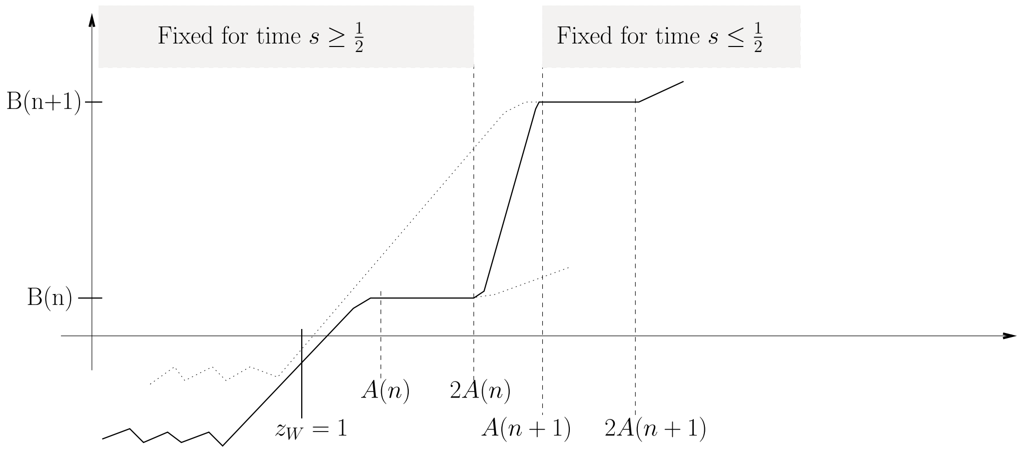

Fig. 9: The homotopy $H_{n,n+1}$ at time $s = \frac{1}{2}$

Corollary 93. $SH_*^{>0}(V, \partial W) \simeq SH_*(W)$ *and* $SH^*_{>0}(V, \partial W) \simeq SH^*(W)$.

Proof: We only prove the corollary for homology, cohomology being completely analog. Take the sequence of Hamiltonians (H_n) constructed in Proposition 92. Clearly it is cofinal and $(H_n) \subset Ad^{>0}(V, \partial W)$, as 1-periodic orbits with positive action are either isolated critical points inside W (as H is Morse and C^2-small there) or isolated Reeb-orbits near $z_W = 1$ – in both cases non-degenerate. Hence we have

$$SH_*^{>0}(V, \partial W) = \varinjlim FH_*^{>0}(H_n).$$

Let $\tilde{H}_n \in Ad(W)$ be the linear extension of $H_n|_W$ with slope $k(n)$. Due to $k(n) \notin Spec(\partial W, \lambda)$, we have obviously $FC_*^{>0}(H_n) = FC_*(\tilde{H}_n)$. As any $\mathcal{A}^H$-gradient trajectory connecting 1-periodic orbits in W stays in W, the two boundary operators ∂^{H_n} and $\partial^{\tilde{H}_n}$ coincide and we have $FH_*^{>0}(H_n) = FH_*(\tilde{H}_n)$. As the $\mathcal{A}^H$-gradient trajectories for the homotopies $H_{n,n+1}$ stay inside W, the continuation maps

$$\sigma(H_{n+1}, H_n) : FH_*^{>0}(H_n) \to FH_*^{>0}(H_{n+1})$$

coincide with the continuation maps

$$\sigma(\tilde{H}_{n+1}, \tilde{H}_n) : FH_*(\tilde{H}_n) \to FH_*(\tilde{H}_{n+1}).$$

Hence we have $\quad SH_*^{>0}(V, \partial W) = \varinjlim FH_*^{>0}(H_n) = \varinjlim FH_*(\tilde{H}_n) = SH_*(W).$ $\square$

We finish this section with the following Invariance Theorem due to Cieliebak, [12].

Theorem 94 (Invariance of SH under subcritical surgery)**.**
Let W be an exact Liouville domain and let V be obtained from W by attaching to $\partial W \times [0,1]$ a subcritical exact symplectic handle H_k^{2n}, $k < n$, as described in Section 5. Moreover, assume that the Conley-Zehnder index is well-defined on W. Then it holds that

$$SH_*(V) \cong SH_*(W) \qquad \textit{and} \qquad SH^*(V) \cong SH^*(W).$$

Proof: The idea of the proof is to construct yet another cofinal sequence of Hamiltonians $(H_n) \subset Ad^w(V) \cap Ad(V, \partial W)$ for which we can directly show that

$$\begin{aligned}
SH_*(W) &\overset{(*)}{\cong} SH_*^{>0}(V, \partial W) &&\overset{(1)}{=} \lim_{n\to\infty} FH_*^{>0}(H_n) \overset{(2)}{\simeq} \lim_{n\to\infty} FH_*(H_n) \overset{(3)}{=} SH_*(V) \\
SH^*(W) &\overset{(*)}{\cong} SH^*_{>0}(V, \partial W) &&\overset{(4)}{=} \lim_{n\to\infty} FH^*_{>0}(H_n) \overset{(5)}{\simeq} \lim_{n\to\infty} FH^*(H_n) \overset{(6)}{=} SH^*(V).
\end{aligned}$$

Note that the isomorphisms $(*)$ have been shown in Corollary 93.

To start, fix sequences $k(n) \notin Spec(\partial W, \lambda)$, $k(n) \to \infty$ and $\varepsilon(n) \to 0$. Then choose an increasing sequence of non-degenerate Hamiltonians H_n on W that is on $\partial W \times (-\varepsilon(n), 0]$ of the form

$$H_n|_{\partial W \times (-\varepsilon(n),0]} = k(n) \cdot e^r - \big(1 + \varepsilon(n)\big)$$

and extend H_n over the handle by a function ψ with $\alpha = k(n)$ and $\beta = -1 - \varepsilon(n)$ as described in Section 5.
For each n choose the handle so thin such that each trajectory of X_{H_n} which leaves and reenters the handle has length greater than 1. Thus we obtain a cofinal weakly admissible sequence (H_n), whose 1-periodic orbits having positive action are all contained in W. This shows already the identities $(1), (3), (4)$ and (6). Now recall that we had the long exact sequences

$$\begin{aligned}
\cdots \to FH_{j+1}^{>0}(H_n) \to FH_j^{\le 0}(H_n) \to FH_j(H_n) \to FH_j^{>0}(H_n) \to \dots \\
\cdots \to FH^{j-1}_{\le 0}(H_n) \to FH^j_{>0}(H_n) \to FH^j(H_n) \to FH^j_{\le 0}(H_n) \to \dots
\end{aligned}$$

Note that $FH_j^{>0}(H_n)$ is generated by all 1-periodic orbits of H_n inside W, while $FH_j^{\leq 0}(H_n)$ is generated by all other orbits. These are finitely many, lying on the handle, and are explicitly given in (51). Observe that H_n is on the handle time-independent. The orbits there are therefore of Morse-Bott type. We can now use either the definition of SH with Morse-Bott techniques, as described in [7], or perturb H_n near these orbits to make it non-degenerate, as described in [13]. In both cases we obtain for each orbit γ two generators in the chain complex whose indices are $\mu_{CZ}(\gamma)$ and $\mu_{CZ}(\gamma)+1$. We have shown in Section 5.4 that the possible values of $\mu_{CZ}(\gamma)$ increase to ∞ as the slope $\alpha = k(n)$ tends to ∞. Therefore, $FH_j^{\leq 0}(H_n)$ becomes eventually zero for n large enough, as well as $FH_{j+1}^{\leq 0}(H_n)$. This implies for n large enough that

$$FH_j(H_n) \to FH_j^{>0}(H_n)$$

is an isomorphism. As the direct limit is an exact functor, these maps converge to an isomorphism in the limit, proving (2). In the cohomology case, the line of arguments is the same. Even though taking inverse limits is not exact, it still takes the isomorphism

$$FH^j_{>0}(H_n) \to FH^j(H_n)$$

to an isomorphism in the limit, as it is left exact (see Theorem 73). This proves (5). $\square$

6.6. Rabinowitz-Floer and symplectic (co)homology

In [16], Cieliebak, Frauenfelder and Oancea showed that Rabinowitz-Floer homology and symplectic (co)homology are closely related. More precisely, they showed, under the assumption that all homologies are $\mathbb{Z}$-graded by the Conley-Zehnder index, that there exists the following exact sequence:

$$... \to SH^{-*}(V) \to SH_*(V) \to RFH_*(V) \to SH^{-(*-1)}(V) \to SH_{*-1} \to ... \qquad (55)$$

where the map $SH^{-*}(V) \to SH_*(V)$ fits into the commutative diagram

$$\begin{tikzcd} SH^{-*}(V) \arrow[r] \arrow[d, "c^*"'] & SH_*(V) \\ H^{-*+n}(V,\partial V) \arrow[r] & H_{*+n}(V,\partial V). \arrow[u, "c_*"'] \end{tikzcd}$$

Here, the bottom arrow is the composition of the map induced by the inclusion $i : V \hookrightarrow (V, \partial V)$ together with the Poincaré duality isomorphism

$$H^{-*+n}(V,\partial V) \xrightarrow{PD} H_{*+n}(V) \xrightarrow{i_*} H_{*+n}(V,\partial V).$$

Moreover, there are the following commutative diagrams of long exact sequences, where PD is the Poincaré duality and the top sequence is the (co)homological long exact sequence of the pair $(V, \partial V)$:

$$\begin{tikzcd}
\arrow[r] & H_{*+n}(V) \arrow[r] \arrow[d, equal, "PD"] & H_{*+n}(V,\partial V) \arrow[r] \arrow[d] & H_{*-1+n}(\partial V) \arrow[r] \arrow[d] & H_{*-1+n}(V) \arrow[r] \arrow[d, equal, "PD"] & {} \\
\arrow[r] & H^{-*+n}(V,\partial V) \arrow[r] & SH_*(V) \arrow[r] & RFH_*^{\geq 0}(V) \arrow[r] & H^{-(*-1)+n}(V,\partial V) \arrow[r] & {}
\end{tikzcd} \tag{56}$$

and

$$\begin{tikzcd}
\arrow[r] & H^{-*+n}(V,\partial V) \arrow[r] & H^{-*+n}(V) \arrow[r] & H^{-*+n}(\partial V) \arrow[r] & H^{-(*-1)+n}(V,\partial V) \arrow[r] & {} \\
\arrow[r] & SH^{-*}(V) \arrow[r] \arrow[u] & H_{*+n}(V,\partial V) \arrow[r] \arrow[u, equal, "PD"] & RFH_*^{\leq 0}(V) \arrow[r] \arrow[u] & SH^{-(*-1)}(V) \arrow[r] \arrow[u] & {}.
\end{tikzcd} \tag{57}$$

In the sequence (55), we find in particular for $* \geq n$ or $* \leq -n$ that $SH^{-*}(V) \to SH_*(V)$ is zero. This implies for field coefficients (for example $\mathbb{Z}_2$) and $* \in \mathbb{Z} \setminus [-n+1, n]$ that

$$RFH_*(V) \cong SH_*(V) \oplus SH^{-*+1}(V). \tag{58}$$

The sequences (56) and (57) on the other hand imply for $|\,*\,| \geq n+1$ that

$$SH_*(V) \cong RFH_*^{\geq 0}(V) \qquad \text{and} \qquad SH^{-(*-1)}(V) \cong RFH_*^{\leq 0}(V).$$

In view of the Invariance Theorem 94 for symplectic (co)homology, we obtain from (58) the Invariance Theorem for Rabinowitz-Floer homology.

Theorem 95 (Invariance of RFH under subcritical surgery)**.**
Let W be a Liouville domain and let V be obtained from W by attaching to $\partial W \times [0,1]$ a subcritical exact symplectic handle H_k^{2n}, $k < n$, as described in Section 5. Moreover, assume that the Conley-Zehnder index is well-defined on W. Then it holds for field coefficients that

$$RFH_*(V) \cong RFH_*(W),$$

at least for $* \in \mathbb{Z} \setminus [-n, n+1]$, i.e. away from the singular homology of $(W, \partial W)$.

Remark.

- The restriction to $* \in \mathbb{Z} \setminus [-n+1, n]$ is technical. It is not clear at the moment what happens for $* \in [-n+1, n]$ but it is conjectured that RFH is invariant there as well, just like SH^* and SH_*.

- The use of field coefficients is also technical. However, we cannot drop this assumption, as our proof relies on the direct sum decomposition (58), which itself depends on a splitting of the exact sequence

$$0 \to SH_*(V) \to RFH(V)_* \to SH^{-*+1}(V) \to 0.$$

  Such a splitting exists in general only for field (or semi-simple) coefficients.

- In [17], Rem. 9.15, Cieliebak and Oancea recently proved Theorem 95 directly in the Rabinowitz-Floer setting resp. the isomorphic setting of symplectic homology of trivial cobordisms. This approach avoids the two problems mentioned above.

7. Brieskorn manifolds and exotic contact structures

In this section, we prove our Main Theorem 114 and show some consequences. First, we introduce the Brieskorn manifolds Σ_a with their canonical contact structure. We give explicit exact contact fillings and calculate for some Σ_a their Rabinowitz-Floer homology. Among these manifolds, there are many that are homeomorphic/diffeomorphic to the standard sphere. We use these to construct new contact structures on manifolds which support fillable contact structures.

7.1. Brieskorn manifolds

In this subsection, we recall the construction of Brieskorn manifolds and their contact structures and fillings. It is a shortened version of similar sections in [23] and my diploma thesis. We include it here for completeness and readability.

Let $a = (a_0, a_1, \ldots, a_n)$ be a vector of natural numbers with $a_i \geq 2$ and define a complex polynomial $f \in C^\infty(\mathbb{C}^{n+1})$ by

$$f(z) = z_0^{a_0} + z_1^{a_1} + \ldots + z_n^{a_n}.$$

The next lemma shows that its level sets $V_a(t) := f^{-1}(t)$ are smooth complex hypersurfaces except for $V_a(0)$, which has an isolated singularity at zero. The links of this singularity $\Sigma_a := V_a(0) \cap S^{2n+1}$ are the ***Brieskorn manifolds***.

Lemma 96 (cf. [29] or Fauck, diploma thesis)**.**
The sets Σ_a and $V_a(t)$, $t \neq 0$, are smooth manifolds.

Proof: We set $\rho(z) := ||z||^2 = \sum z_k \bar{z}_k$ and consider the maps

$$f : \mathbb{C}^{n+1} \to \mathbb{C} \qquad \text{and} \qquad (f, \rho) : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{R}.$$

As $V_a(t) = f^{-1}(t)$ and $\Sigma_a = (f, \rho)^{-1}(0, 1)$, it suffices to show that t resp. $(0, 1)$ are regular values. Using Wirtinger calculus, we obtain that

$$D(f, \rho) = \begin{pmatrix} a_0 z_0^{a_0 - 1} & \ldots & a_n z_n^{a_n - 1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & a_0 \bar{z}_0^{a_0 - 1} & \ldots & a_n \bar{z}_n^{a_n - 1} \\ \bar{z}_0 & \ldots & \bar{z}_n & z_0 & \ldots & z_n \end{pmatrix}.$$

For $z \neq 0$, we find that the first two rows of this matrix are linear independent, which shows that $t \neq 0$ is a regular value of f. If $D(f, \rho)$ has not rank 3 and $z \neq 0$, then there exists $\lambda \neq 0$ such that $\bar{z}_k = \lambda a_k z_k^{a_k - 1}$ for all k. Then, we find

$$0 < \sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} = \lambda \sum_{k=0}^{n} z_k^{a_k} = \lambda \cdot f(z),$$

which is impossible for $z \in \Sigma_a \subset f^{-1}(0)$. $\square$

Let us consider on $\mathbb{C}^{n+1}$ the following a-weighted Hermitian form given by

$$\langle \xi, \zeta \rangle_a := \frac{1}{2} \sum_{k=0}^{n} a_k \xi_k \bar{\zeta}_k.$$

It defines an a-weighted symplectic 2-form $\omega_a = -\mathfrak{Im}\langle \cdot, \cdot \rangle_a$, explicitly given by

$$\omega_a := \frac{i}{4} \sum_{k=0}^{n} a_k dz_k \wedge d\bar{z}_k.$$

Note that $Y_\lambda(z) := z/2$ is a Liouville vector field for ω_a, whose Liouville 1-form is

$$\lambda_a := \omega_a(Y_\lambda, \cdot) = \frac{i}{8} \sum_{k=0}^{n} a_k (z_k d\bar{z}_k - \bar{z}_k dz_k).$$

Proposition 97 (Lutz & Meckert, [34]).
The restriction $\alpha_a := \lambda_a|_\Sigma$ is a contact form on Σ_a with Reeb vector field R_a given by

$$R_a = 4i \left(\frac{z_0}{a_0}, \dots, \frac{z_n}{a_n} \right).$$

Proof: The gradient $\nabla_a f$ of f with respect to $\langle \cdot, \cdot \rangle_a$ if given by

$$\nabla_a f := 2 \left(\bar{z}_0^{a_0 - 1}, \dots, \bar{z}_n^{a_n - 1} \right).$$

The Liouville vector field Y_V of the restricted 1-form $\lambda_a|_{V_a(0)}$ with respect to the restricted symplectic form $\omega_a|_{V_a(0)}$ is given by

$$Y_V := Y_\lambda - \frac{\langle \nabla_a f, Y_\lambda \rangle_a}{||\nabla_a f||_a^2} \cdot \nabla_a f.$$

Indeed, $TV_a(t) = \ker df = \ker \langle \nabla_a f, \cdot \rangle_a$, which shows that $Y_V \in TV_a(0)$. Moreover, we calculate for any $\xi \in TV_a(0)$

$$\omega_a(Y_V, \xi) = \omega_a(Y_\lambda, \xi) - \frac{\langle \nabla_a f, Y_\lambda \rangle_a}{||\nabla_a f||_a^2} \omega_a(\nabla_a f, \xi) = \lambda_a(\xi) + \frac{\langle \nabla_a f, Y_\lambda \rangle_a}{||\nabla_a f||_a^2} \underbrace{\mathfrak{Im}\langle \nabla_a f, \xi \rangle_a}_{=0} = \lambda_a(\xi).$$

This shows that Y_V is the Liouville vector field for the pair $(\omega_a|_{V_a(0)}, \lambda_a|_{V_a(0)})$. Now, note that $d\rho = \sum (\bar{z}_k dz_k + z_k d\bar{z}_z)$ and calculate

$$d\rho(Y_V) = \sum \frac{\bar{z}_k z_k}{2} - \frac{\langle \nabla_a f, Y_\lambda \rangle_a}{||\nabla_a f||_a^2} \sum 2\bar{z}_k \cdot \bar{z}_k^{a_k - 1} = \frac{\rho(z)}{2} - \frac{\langle \nabla_a f, Y_\lambda \rangle_a}{||\nabla_a f||_a^2} \cdot 2\overline{f(z)} = \frac{1}{2} > 0$$

as $\rho(z) = 1$ and $f(z) = 0$ for $z \in \Sigma_a$. It follows that Y_V points out of the unit sphere $S^{2n+1} = \rho^{-1}(0)$ and hence out of Σ_a in $V_a(0)$. We obtain from Lemma 3 that Σ_a is therefore a contact hypersurface in $V_a(0)$.

It remains to check that R_a is the Reeb vector field of α_a. We have for $z \in \Sigma_a$

$$\left.\begin{aligned} \langle R_a, \nabla_a f\rangle_a &= 4i\sum z_k^{a_k} = 4if(z) = 0 \\ d\rho(R_a) &= \sum \bar{z}_k 4iz_k + z_k(-4i)\bar{z}_k = 0 \end{aligned}\right\} \Rightarrow R_a(z) \in T_z\Sigma_a$$

$$\text{and} \qquad \alpha_a(R_a) = \lambda_a(R_a) = \frac{i}{8}\sum a_k \left(z_k\frac{-4i}{a_k}\bar{z}_k - \bar{z}_k\frac{4i}{a_k}z_k\right) = \frac{-i^2 8}{8}\rho(z) = 1.$$

Finally, we calculate for $\xi \in T_z\Sigma_a$ that

$$\begin{aligned} d\alpha_a(R_a,\xi) = \omega_a(R_a,\xi) = \frac{i}{4}\sum a_k\left(4i\frac{z_k}{a_k}\bar{\xi}_k - (-4i)\frac{\bar{z}_k}{a_k}\xi_k\right) &= -\sum\left(z_k\bar{\xi}_k + \bar{z}_k\xi_k\right) \\ &= -d\rho(\xi) = 0. \end{aligned}$$

Hence, R_a is the Reeb vector field of α_a, as $\alpha_a(R_a) = 1$ and $\iota(R_a)d\alpha_a = 0$. $\square$

Corollary 98. *The symplectic complement $\xi_a^\perp$ with respect to ω_a of the contact structure $\xi_a := \ker\alpha_a$ is symplectically trivialized by the following 4 vector fields:*

$$X_1 := \frac{\nabla_a f}{||\nabla_a f||_a}, \quad Y_1 := i\cdot X_1, \quad X_2 := Y_V = Y_\lambda - \frac{\langle\nabla_a f, Y_\lambda\rangle_a}{||\nabla_a f||_a^2}\cdot\nabla_a f, \;\; \textit{and } Y_2 := R_a.$$

Explicitly, X_1 and X_2 are given by

$$\begin{aligned} X_1 &= \sqrt{\frac{2}{\sum a_k|z_k|^{2(a_k-1)}}}\cdot\left(\bar{z}_0^{a_0-1},\dots,\bar{z}_n^{a_n-1}\right), \\ X_2 &= \frac{1}{2}\cdot\left(z_0 - \frac{\sum a_k z_k^{a_k}}{\sum a_k|z_k|^{2(a_k-1)}}\cdot\bar{z}_0^{a_0-1},\dots,z_n - \frac{\sum a_k z_k^{a_k}}{\sum a_k|z_k|^{2(a_k-1)}}\cdot\bar{z}_n^{a_n-1}\right). \end{aligned}$$

Proof: The explicit descriptions of X_1, X_2 are obtained by easy calculations from the definition of $\nabla_a f$ and $\langle\cdot,\cdot\rangle_a$. Note that X_1, Y_1 generate the complex complement of $TV_a(0)$ while X_2, Y_2 generate the symplectic complement of ξ_a in $TV_a(0)$. This shows that

$$\omega_a(X_1,X_2) = \omega_a(X_1,Y_2) = \omega_a(Y_1,X_2) = \omega_a(Y_1,Y_2) = 0.$$

The norming guarantees $\omega_a(X_1,Y_1) = 1$, while $\omega_a(X_2,Y_2) = 1$ follows from the proof of Proposition 97. $\square$

To define the Rabinowitz-Floer homology on Σ_a, we need a Liouville domain (W,λ) with boundary (Σ_a,α_a). Unfortunately, we cannot take $V_a(0)\cap B_1(0)$, as it has a singularity at 0. We overcome this obstacle by constructing an interpolation between $V_a(0)$ and $V_a(\varepsilon)$ for $\varepsilon > 0$ small. To do this, choose a smooth monotone decreasing cut-off function $\beta \in C^\infty(\mathbb{R})$ with $\beta(x) = 1$ for $x \leq 1/4$ and $\beta(x) = 0$ for $x \geq 3/4$. Then define

$$\begin{aligned} V_\varepsilon &:= \left\{z\in\mathbb{C}^{n+1}\,\middle|\, z_0^{a_0} + z_1^{a_1} + \dots + z_n^{a_n} = \varepsilon\cdot\beta\big(||z||^2\big)\right\} \\ \text{and} \quad W_\varepsilon &:= V_\varepsilon \cap B_1(0). \end{aligned}$$

Proposition 99. *(V_ε, λ) is a Liouville domain with boundary (Σ_a, α_a) and vanishing first Chern class $c_1(TV)$, if ε is small enough.*

Proof:

- Consider on $\mathbb{C}^{n+1}$ the smooth complex valued function $f_\varepsilon(z) := f(z) - \varepsilon \cdot \beta\big(||z||^2\big)$. Its differential is given by

$$Df_\varepsilon = Df - \varepsilon \cdot \beta'\big(||z||^2\big) \cdot D\big(||z||^2\big).$$

  As Df is non-degenerate for $z \neq 0$, it follows that 0 is a regular value of f_ε for ε small enough, as $0 \notin f_\varepsilon^{-1}(0)$ and $\beta'\big(||z||^2\big) \neq 0$ only for $1/4 \leq ||z||^2 \leq 3/4$. We find hence that $V_\varepsilon = f_\varepsilon^{-1}(0)$ is a smooth manifold.

- Next, we note that the two functions $\mathfrak{Re}(f_\varepsilon)$ and $\mathfrak{Im}(f_\varepsilon)$ have with respect to ω_a the following Hamiltonian vector fields

$$X_{\mathfrak{Re} f_\varepsilon} = X_{\mathfrak{Re} f} + \varepsilon \cdot X_{\beta(||z||^2)}, \qquad X_{\mathfrak{Im} f_\varepsilon} = X_{\mathfrak{Im} f},$$

  as $\beta\big(||z||^2\big)$ is a real valued function. Moreover, it follows from the fact

$$T_z V_\varepsilon = \big\{\xi \in T\mathbb{R}^{2n+2} \,\big|\, \omega_a(\xi, X_{\mathfrak{Re} f_\varepsilon}) = 0 \text{ and } \omega_a(\xi, X_{\mathfrak{Im} f_\varepsilon}) = 0\big\}$$

  that *span* $\{X_{\mathfrak{Re} f_\varepsilon}(z), X_{\mathfrak{Im} f_\varepsilon}(z)\}$ is the ω_a-symplectic complement of $T_z V_\varepsilon$. As $X_{\mathfrak{Re} f}$ and $X_{\mathfrak{Im} f}$ span the complex complement of $T_z V$, we find that $\omega_a(X_{\mathfrak{Re} f}, X_{\mathfrak{Im} f}) \neq 0$ and therefore that for ε small enough holds

$$\omega_a(X_{\mathfrak{Re} f_\varepsilon}, X_{\mathfrak{Im} f_\varepsilon}) = \omega_a(X_{\mathfrak{Re} f}, X_{\mathfrak{Im} f}) + \varepsilon \cdot \omega_a(X_{\beta(||z||^2)}, X_{\mathfrak{Im} f}) \neq 0.$$

  This implies that *span* $\{X_{\mathfrak{Re} f_\varepsilon}(z), X_{\mathfrak{Im} f_\varepsilon}(z)\}$ is a symplectic subspace of $T\mathbb{R}^{2n+2}$ and hence that its symplectic complement TV_ε is also a symplectic subspace, in other words $\omega_a|_{TV_\varepsilon}$ is non-degenerate. As $\omega_a|_{TV_\varepsilon} = d\lambda_a|_{TV_\varepsilon}$, we know that $(V_\varepsilon, \lambda_a|_{TV_\varepsilon})$ is an exact symplectic manifold and that $(W_\varepsilon, \lambda_a|_{TV_\varepsilon})$ is a Liouville domain.

- Finally, V_ε is diffeomorphic to $V_a(\varepsilon)$ as both are diffeomorphic to the set

$$V_a(\varepsilon) \cap B_{1/2} = \{z \in V_a(\varepsilon)\ :\ ||z|| < 1/2\} = V_\varepsilon \cap B_{1/2}.$$

  To see this, consider on V_ε and $V_a(\varepsilon)$ the function $\rho(z) = ||z||^2$, whose critical points lie for V_ε and $V_a(\varepsilon)$ in $V_\varepsilon \cap B_{1/2}$ if ε is small enough. It follows from classical Morse-theory that V_ε and $V_a(\varepsilon)$ are hence diffeomorphic to $V_\varepsilon \cap B_{1/2}$. Since $V_a(\varepsilon)$ is parallelizable, V_ε is parallelizable as well and hence $c_1(TV) = 0$ (for more details see [29], § 14). $\square$

Discussion 100. On Σ_a, we have symplectic symmetries σ of the form

$$\sigma(z) = \big(c_0 \cdot z_0,\ \dots\ , c_n \cdot z_n\big), \qquad c_k \in \sqrt[a_k]{1} \subset \mathbb{C},$$

where the c_k are complex a_k-roots of 1. These symmetries extend to V_ε, as

$$0 = \sum z_k^{a_k} - \phi\big(||z||^2\big) = \sum \big(c_k z_k\big)^{a_k} - \phi\left(\sum ||c_k z_k||^2\right).$$

The Reeb vector field $R_a = 4i\big(z_0/a_0, \dots, z_n/a_n\big)$ generates the following flow:

$$\varphi_a^t(z) = \left(e^{4it/a_0} \cdot z_0,\ \dots,\ e^{4it/a_n} \cdot z_n\right). \tag{59}$$

The sets of closed Reeb orbits of periodic η are found by critical inspection as

$$\mathcal{N}^\eta = \left\{ z \in \Sigma_a \;\middle|\; z_k = 0 \quad \text{if} \quad \frac{4\eta}{a_k} \notin 2\pi\mathbb{Z} \right\}.$$

Note that $\mathcal{N}^\eta = \emptyset$, if $4\eta/a_k \in 2\pi\mathbb{Z}$ does not hold for at least two different k, as vectors $z \in \Sigma_a$ have at least two non-zero entries. To ease the notation, we define the integer $L := 2\eta/\pi$, so that

$$\mathcal{N}^\eta = \mathcal{N}^{L\pi/2} = \left\{ z \in \Sigma_a \;\middle|\; z_k = 0 \quad \text{if} \quad \frac{L}{a_k} \notin \mathbb{Z} \right\}. \tag{60}$$

Note that $\mathcal{N}^{L\pi/2}$ is the intersection $\Sigma_a \cap E(a, L)$ of Σ_a with the complex linear subspace $E(a, L) \subset \mathbb{C}^{n+1}$ given by

$$E(a, L) := \left\{ z \in \mathbb{C}^{n+1} \;\middle|\; z_k = 0 \quad \text{if} \quad L/a_k \notin \mathbb{Z} \right\}.$$

Its complex dimension is given by

$$\dim_{\mathbb{C}} E(a, L) = n(a, L) := \#\left\{ k \;\middle|\; 0 \le k \le n \text{ and } L/a_k \in \mathbb{Z} \right\}.$$

We find that $\mathcal{N}^{L\pi/2}$ is therefore isomorphic to the Brieskorn manifold $\Sigma_{a(L)}$ in $\mathbb{C}^{n(a,L)} \cong E(a, L)$ where

$$a(L) = \big(a_{k_1}, \dots, a_{k_{n(a,L)}}\big) \subset \big(a_0, \dots, a_n\big) = a$$

is the subvector of a defined by $a_{k_i} \in a(L)$ if and only if $L/a_{k_i} \in \mathbb{Z}$. The 1-form $\alpha_a|_{T\mathcal{N}^{L\pi/2}}$ is hence isomorphic to the contact form $\alpha_{a(L)}$.
The differential of φ_a at time $L\pi/2$ is given by

$$D\varphi_a^{L\pi/2} = diag\left(e^{2\pi i L/a_0}, \dots, e^{2\pi i L/a_n}\right).$$

It follows that

$$\ker\big(D_z\varphi^{L\pi/2}|_{T_z\Sigma_a} - Id\big) = T_z\Sigma_a \cap E(a, L) = T_z\mathcal{N}^{L\pi/2}.$$

We have thus proven the following proposition.

Proposition 101. *All sets $\mathcal{N}^{L\pi/2}$ of closed Reeb orbits on (Σ_a, α_a) satisfy (MB).*

Next, we give some topological facts about Brieskorn manifolds, as shown by Egbert Brieskorn in [10]. To give them as precisely as possible, we introduce for every tuple $a = (a_0, \dots, a_n)$ the following graph G_a:

- G_a has $n + 1$ vertices labeled $a_0, \dots, a_n$.
- G_a contains an edge between a_j, a_k if and only if $gcd(a_j, a_k) > 1$.

Let K be the connected subgraph of G that consists of all even a_k. We say that G_a satisfies condition (O) if

$$|K| \textit{ is odd and for all } a_j, a_k \in K \textit{ holds } gcd(a_j, a_k) = 2. \tag{O}$$

Theorem 102 (Brieskorn, cf. [10])**.** *Every Brieskorn manifold Σ_a satisfies*

i. Σ_a is at least $(n-2)$-connected, i.e. $\pi_k(\Sigma_a) = 0$, $1 \leq k \leq n-2$, which implies in particular for the singular homology of Σ_a that

$$H_k(\Sigma_a) = 0 \qquad \textit{for} \quad k \neq 0, n-1, n, 2n-1.$$

ii. Σ_a is homeomorphic to the sphere S^{2n-1} if and only if G_a contains two isolated vertices or G_a has one isolated vertex and satisfies (O).

Remark.

- Given the tuples $(a_0, \dots, a_n)$, it is possible to calculate $H_{n-1}(\Sigma_a) \cong H_n(\Sigma_a)$ (see for example [32]).
- It follows from $\pi_1(\Sigma_a) = 0$ and $c_1(TV_\varepsilon) = 0$ that $(V_\varepsilon, \Sigma_a)$ satisfies conditions (A) and (B), i.e. that Conley-Zehnder indices on $(V_\varepsilon, \Sigma_a)$ are well-defined.

We conclude this subsection with the calculation of the indices of all closed Reeb orbits in $\mathcal{N}^{L\pi/2}$. This is mainly taken from my diploma thesis and follows [32] and [51].
Recall the definitions of R_a (Proposition 97) and its flow φ_a^t in (59). We can obviously regard them as defined on $\mathbb{C}^{n+1}$ instead of Σ_a. This allows us to calculate indices directly on $T\mathbb{C}^{n+1} = \left(\mathbb{C}^{n+1}\right)^2$ instead of $T\Sigma_a$. The action of $D\phi_a^t$ on $T\mathbb{C}^{n+1}$ in terms of the standard trivialization is given by the following path $\Phi^t \in Sp(2n+2)$ of diagonal matrices:

$$D\phi_a^t = diag\left(e^{4it/a_0}, \dots, e^{4it/a_n}\right) =: \Phi^t.$$

Recall that Corollary 98 gave the following symplectic trivialization of the symplectic complement $\xi_a^\perp$ of ξ_a:

$$\begin{aligned}
X_1 &= \sqrt{\frac{2}{\sum a_k |z_k|^{2(a_k-1)}}} \cdot \left(\bar{z}_0^{a_0-1}, \dots, \bar{z}_n^{a_n-1}\right), \qquad Y_1 = i \cdot X_1,\\
X_2 &= \frac{1}{2} \cdot \left(z_0 - \frac{\sum a_k z_k^{a_k}}{\sum a_k |z_k|^{2(a_k-1)}} \cdot \bar{z}_0^{a_0-1}, \dots, z_n - \frac{\sum a_k z_k^{a_k}}{\sum a_k |z_k|^{2(a_k-1)}} \cdot \bar{z}_n^{a_n-1}\right),\\
Y_2 &= \left(\frac{4i}{a_0} z_0, \dots, \frac{4i}{a_n} z_n\right).
\end{aligned}$$

We find by some calculation that the action of ϕ_a^t on $\xi_a^\perp$ yields

$$\begin{aligned}
D\varphi_a^t\big(X_1(z)\big) &= e^{4it} \cdot X_1\big(\varphi_a^t(z)\big), & D\varphi_a^t\big(Y_1(z)\big) &= e^{4it} \cdot Y_1\big(\varphi_a^t(z)\big),\\
D\varphi_a^t\big(X_2(z)\big) &= X_2\big(\varphi_a^t(z)\big), & D\varphi_a^t\big(Y_2(z)\big) &= Y_2\big(\varphi_a^t(z)\big).
\end{aligned}$$

It follows that the action of $D\phi_a^t$ on $\xi_a^\perp$ in the trivialization given by X_1, Y_1, X_2, Y_2 is the following path $\Phi_2^t \in Sp(4)$ of diagonal matrices:

$$diag(e^{4it}, 1) =: \Phi_2^t.$$

Observe that Φ^t and Φ_2^t are linearizations of φ_a^t on $T\mathbb{C}^{n+1}$ and $\xi_a^\perp$ respectively, which are both trivial bundles. A trivialization of ξ_a over a disc $u \subset \Sigma_a$, with $\partial u = v$ being a Reeb trajectory, provides us with a linearization $\Phi_1^t \in Sp(2n-2)$ of φ_a^t on ξ_a. Any trivialization of ξ_a over any disc in Σ_a followed by the above trivialization of $\xi_a^\perp$ gives again a trivialization of $T\mathbb{C}^{n+1}$, which is homotopic to the standard one, as the disc is contractible. We hence obtain that $\Phi^t = \Psi^t(\Phi_1^t \oplus \Phi_2^t)(\Psi^0)^{-1}$ for some contractible loop $\Psi \in Sp(2n+2)$.

Now, let $v \in \mathcal{N}^{L\pi/2}$ be a closed Reeb trajectory of length $L\pi/2$. Using Lemma 59 and the product property of the Conley-Zehnder index, we find that

$$\begin{aligned}\mu_{CZ}(v) = \mu_{CZ}(\Phi_1) = \mu_{CZ}(\Phi) - \mu_{CZ}(\Phi_2) &= \underbrace{\sum_{k=0}^{n}\left(\left\lfloor \frac{L}{a_k}\right\rfloor + \left\lceil \frac{L}{a_k}\right\rceil\right)}_{\Phi-comp.} - \underbrace{(\lfloor L\rfloor + \lceil L\rceil)}_{\Phi_2-comp.} \\ &= \sum_{k=0}^{n}\Big(\lfloor L/a_k\rfloor + \lceil L/a_k\rceil\Big) - 2L,\end{aligned}$$

where the last line holds as L is an integer.

We have shown above that the manifold $\mathcal{N}^{L\pi/2}$ is isomorphic to the Brieskorn manifold $\Sigma_a \cap E(a, L)$ with $\dim E(a, L) = 2n(a, L)$. Its dimension is thus

$$\dim \mathcal{N}^{L\pi/2} = 2 \cdot n(a, L) - 3 = 2 \cdot \#\Big\{k \,\Big|\, L/a_k \in \mathbb{Z}\Big\} - 3.$$

Note that $(n + 1) - n(a, L)$ is the number of indices k, where $\lceil L/a_k\rceil - \lfloor L/a_k\rfloor = +1$. Thus, we find that

$$\begin{aligned}\mu_{CZ}(v) - \frac{1}{2}\dim \mathcal{N}^{L\pi/2} &= \sum_{k=0}^{n}\left(\left\lfloor \frac{L}{a_k}\right\rfloor + \left\lceil \frac{L}{a_k}\right\rceil\right) - 2L - \frac{2n(a, L) - 3}{2} \\ &= \sum_{k=0}^{n}\left(\left\lfloor \frac{L}{a_k}\right\rfloor + \left\lceil \frac{L}{a_k}\right\rceil\right) + (n+1) - n(a, L) - 2L - (n+1) + \frac{3}{2} \\ &= \sum_{k=0}^{n} 2 \cdot \left\lceil \frac{L}{a_k}\right\rceil - 2L - (n-1) - \frac{1}{2}.\end{aligned}$$

Using the definition of the index μ (cf. Proposition 61), we have shown the following:

Proposition 103. *Let h be a Morse function on $crit\big(\mathcal{A}^H\big)$ and $c = (v, \eta) \in \mathcal{N}^\eta \cap crit(h)$ with $\eta = L\pi/2$ and v a closed Reeb orbit of length η. The index of c is then given by*

$$\mu(c) = 2 \cdot \left(\sum_{k=0}^{n}\left\lceil \frac{L}{a_k}\right\rceil\right) - 2L + ind_h(c) - (n-1).$$

Discussion 104. As $\dim \mathcal{N}^{L\pi/2} = 2n(a,L) - 3 \leq 2n-1$, we find that $ind_h(c)$ lies in the interval $[0, 2n-1]$ for every $c \in crit(h)$. Set $A := \prod_{k=0}^n a_k$ and write $L = j \cdot A + l$ with $l \in [0, A-1]$. Then we have

$$\begin{aligned}
\mu(c) &= 2 \cdot \left(\sum_{k=0}^{n} \left\lceil \frac{L}{a_k} \right\rceil \right) - 2L + ind_h(c) - (n-1) \\
&= 2j \left(\sum_{k=0}^{n} \frac{A}{a_k} - A \right) + 2 \cdot \left(\sum_{k=0}^{n} \left\lceil \frac{l}{a_k} \right\rceil \right) - 2l + ind_h(c) - (n-1) \\
&=: j \cdot 2A \left(\sum_{k=0}^{n} \frac{1}{a_k} - 1 \right) + D(l,c),
\end{aligned}$$

where $D(l,c)$ depends on l and c but $-2A - n \leq D(l,c) \leq 2nA + n$. Thus we find

- If $\sum 1/a_k > 1$, then $\mu \to \infty$ as $L \to \infty$.
- If $\sum 1/a_k < 1$, then $\mu \to -\infty$ as $L \to \infty$.
- If $\sum 1/a_k = 1$, then μ is uniformly bounded.

As the set $crit(h)$ generates $RFH(W, \Sigma_a)$ for any filling W of Σ_a, we find the following:

Corollary 105. *For every Brieskorn manifold Σ_a and every filling W of Σ_a holds that $RFH_*(W, \Sigma_a)$ is a finite dimensional $\mathbb{Z}_2$-vector space for all $*$ if $\sum 1/a_k \neq 1$. If $\sum 1/a_k = 1$, it is zero for almost all $*$, with the exceptions lying in $[-2A-n, 2nA+n]$. (Actually, (62) and (63) show that non-zero groups can only occur for $* \in [-n+1, n]$.)*

7.2. Calculation of $RFH(W_\varepsilon, \Sigma_a)$ for some a

Originally, the intention of this work was to calculate $RFH(W_\varepsilon, \Sigma_a)$ for all a. By now, we are still far from achieving this goal. Hence, we restrict ourself to some subclasses of Brieskorn manifolds where calculations are doable and which are interesting in its own.

We start with the last case of Discussion 104, i.e. we assume that $\sum_{k=0}^n 1/a_k = 1$. First we assume all a_k to be equal, i.e. we consider

$$a = \big(n+1, n+1, \dots, n+1\big).$$

Here, the Reeb flow φ_a (cf. (59)) is given by $\varphi_a^t(z) = e^{4it/(n+1)} \cdot z$. We find that the critical manifolds of closed Reeb orbits $\mathcal{N}^{L\pi/2}$ are non-trivial exactly if $L = l(n+1)$ for some $l \in \mathbb{Z}$. Moreover, all these manifolds are equal to Σ_a. Their singular homology groups with $\mathbb{Z}_2$-coefficients $H_*(\mathcal{N}^{l(n+1)\pi/2}, \mathbb{Z}_2)$ are hence by Theorem 102 non-zero only for $* = 0, n-1, n, 2n-1$. Now let

- $l\gamma_0$ denote *the* generator of $H_0(\mathcal{N}^{l(n+1)\pi/2}, \mathbb{Z}_2)$,
- $l\gamma_{n-1}^j$ denote the generators of $H_{n-1}(\mathcal{N}^{l(n+1)\pi/2}, \mathbb{Z}_2)$,
- $l\gamma_n^j$ denote the generators of $H_n(\mathcal{N}^{l(n+1)\pi/2}, \mathbb{Z}_2)$,
- $l\gamma_{2n-1}$ denote *the* generator of $H_{2n-1}(\mathcal{N}^{l(n+1)\pi/2}, \mathbb{Z}_2)$.

By Section 4.3, Theorem 85, we know that we can calculate the homology groups $RFH_*(W_\varepsilon, \Sigma_a)$ via a chain complex $RFC_*(V_\varepsilon, \Sigma)$ generated by the elements $l\gamma_k^j$ (or equivalently, we can pretend that there is a perfect Morse function h on $\mathcal{N}^{l(n+1)\pi/2}$). The index $\mu(l\gamma_k^j)$ is due to Proposition 103 given by

$$\begin{aligned}
\mu(l\gamma_0) &= \underbrace{2 \cdot \left(\sum_{k=0}^{n} \left\lceil \frac{l(n+1)}{n+1} \right\rceil \right) - 2l(n+1)}_{=0} + \underbrace{0}_{=ind_h(l\gamma_0)} - (n-1) &&= -n+1 \\
\mu(l\gamma_{n-1}^j) &= 0 + (n-1) - (n-1) &&= 0 \\
\mu(l\gamma_n^j) &= 0 + n - (n-1) &&= 1 \\
\mu(l\gamma_{2n-1}) &= 0 + (2n-1) - (n-1) &&= n.
\end{aligned} \tag{61}$$

Hence $RFC_*(W_\varepsilon, \Sigma_a)$ has for $a = (n+1, \dots, n+1)$ an infinite number of generators in degrees $* = -(n-1), 0, 1$ and n and no generators in all other degrees. From these observations, we obtain directly the following Proposition.

Proposition 106. *Let $a = (n+1, n+1, \dots, n+1)$ and $n \geq 3$. Then $RFH_*(W_\varepsilon, \Sigma_a)$ is independent of the filling and satisfies*

$$\dim_{\mathbb{Z}_2} RFH_*(W_\varepsilon, \Sigma_a) = \begin{cases} \infty & \text{if } * = -n+1 \text{ or } n \\ 0 & \text{if } * \neq -n+1, 0, 1, n. \end{cases}$$

The independence from the filling is obvious for $* \neq 0, 1$ as the chain complex is independent from the filling. For $* = 0, 1$ it follows from Theorem 65 as we can estimate the Conley-Zehnder part of the index μ by

$$\mu_{CZ}(\gamma) = 2 \cdot \left(\sum \left\lceil \frac{L}{a_k} \right\rceil \right) - 2L \geq 2L\Big(\sum 1/a_k - 1 \Big) = 0 > 3 - n.$$

This shows also that $RFH_*(W_\varepsilon, \Sigma_a)$ is independent of the filling whenever $\sum 1/a_k = 1$.

Remark. By now, we have no means to calculate the groups $RFH_*(W_\varepsilon, \Sigma_a)$ for $* = 0, 1$. For that, we would need the operator ∂^F explicitly, as it might be non-zero here.

If not all a_k are equal, we can still get from $\sum 1/a_k = 1$ the following estimates on the index $\mu(\gamma)$ for a generator $\gamma \in H_*(\mathcal{N}^{L\pi/2}, \mathbb{Z}_2)$. If $\lceil L/a_k \rceil \neq L/a_k$, then we have surely $L/a_k < \lceil L/a_k \rceil < L/a_k + 1$. Recalling that $(n+1) - n(a, L)$ is the number of indices with $\lceil L/a_k \rceil \neq L/a_k$ and that the dimension of $\mathcal{N}^{L\pi/2}$ is $2 \cdot n(a, L) - 3$, we get

$$\begin{aligned}
\mu(\gamma) &= 2 \cdot \Big(\sum \lceil L/a_k \rceil \Big) - 2L + * - (n-1) \\
&\leq 2 \cdot \Big(\sum L/a_k + (n+1) - n(a, L) \Big) - 2L + 2 \cdot n(a, L) - 3 - (n-1) \\
&= 2(n+1) - 3 - (n-1) \\
&= n
\end{aligned} \tag{62}$$

$$\text{and} \qquad \begin{aligned}
\mu(\gamma) &\geq 2 \cdot \Big(\sum L/a_k \Big) - 2L + 0 - (n-1) \\
&= -n + 1.
\end{aligned} \tag{63}$$

Note that in these estimates equality can only hold if equality holds for the Conley-Zehnder part, i.e. if $\lceil L/a_k \rceil = L/a_k$ for all k that is if $a_k|L$ for all k. If this is not satisfied, then the estimates (62) and (63) can be sharpened to

$$-(n-1)+2 \leq \mu(\gamma) \leq n-2 \tag{64}$$

as the Conley-Zehnder part of μ is always even. If $a_k|L$ holds for all k, then we have that $\mathcal{N}^{L\pi/2}$ equals Σ_a. In this case we have as above four classes of generators $L\gamma_*^j$ in $RFC(W_\varepsilon, \Sigma_a)$ corresponding to generators of the four non-vanishing singular homology groups $H_*(\mathcal{N}^{L\pi/2}, \mathbb{Z}_2)$, $* = 0, n-1, n, 2n-1$. Note that the index calculations (61) for $l\gamma_*^j$ are also valid for $L\gamma_*^j$. It follows from this observation and the estimates (62), (63), (64) that

$$\dim_{\mathbb{Z}_2} RFC_*(W_\varepsilon, \Sigma_a) = \begin{cases} \infty & \text{if } * = -n+1,\, n \\ 0 & \text{if } * \leq -n,\ * \geq n+1, \quad \text{or} \quad * = -n+2,\, n-1. \end{cases}$$

As before, we hence get:

Proposition 107. *Let $a = (a_0, a_1, \dots, a_n)$ be such that $\sum 1/a_k = 1$. Then it holds that $RFH_*(W_\varepsilon, \Sigma_a)$ is independent from the filling and satisfies*

$$\dim_{\mathbb{Z}_2} RFH_*(W_\varepsilon, \Sigma_a) = \begin{cases} \infty & \textit{if } * = -n+1,\, n \\ 0 & \textit{if } * \leq -n,\ * \geq n+1, \quad \textit{or} \quad * = -n+2,\, n-1. \end{cases}$$

Again, the independence from the filling for $* \leq -n+2$ or $* \geq n-1$ follows already from the specific form of $RFC_*(W_\varepsilon, \Sigma_a)$.

To conclude the case $\sum 1/a_k = 1$, let $A = lcm(a_0, \dots, a_n)$. We find for $* = -n+1$ or n and every $N \in \mathbb{N}$ that there are exactly N generators in $RFC_*(W_\varepsilon, \Sigma_a)$ whose period lies in $(0, NA\pi/2]$. Thus, we find that

$$\dim_{\mathbb{Z}_2} RFH_*^{(0, NA\pi/2]}(W_\varepsilon, \Sigma_a) = N$$

grows linearly in N. A similar result holds for the action window $[-NA\pi/2, 0)$. Using Definition 57 for the growth rates $\Gamma^\pm(W_\varepsilon, \Sigma_a, f)$ we find therefore

Proposition 108. *Let $a = (a_0, \dots, a_n)$ be such that $\sum 1/a_k = 1$. Then it holds for $* = -n+1$ or n that*

$$\Gamma_*^\pm(W_\varepsilon, \Sigma_a, id) = 1.$$

Remark. We observe that the boundedness of the Conley-Zehnder index for all closed Reeb orbits implies that the mean index (see 3.3) for all closed Reeb orbits is zero. This implies in particular that all Brieskorn manifolds Σ_a with $\sum 1/a_k = 1$ are neither Ustilovsky index positive, nor weakly index positive in the sense of Espina, [22], nor have convenient dynamics in the sense of Kwon and van Koert, [33], Defn. 5.14. Using only results known until now, we can therefore not decide whether the contact homology, the S^1-equivariant symplectic homology or the mean Euler characteristic of these examples are invariant under subcritical surgery.

As a second class of Brieskorn manifolds let us now turn to tuples $a = (2, 2, 2, a_3, \dots, a_n)$. In his diploma thesis (see also [23]), the author considered in particular

$$\begin{aligned} & a = (2, \dots, 2, p\quad\ ) \in \mathbb{N}^{n+1}, n \text{ odd}, p \text{ odd} \\ \text{and} \qquad & a = (2, \dots, 2, p, q) \in \mathbb{N}^{n+1}, n \text{ even}, p, q \text{ odd and } gcd(p, q) = 1. \end{aligned}$$

The corresponding Brieskorn manifolds Σ_a are by Theorem 102 homeomorphic to the sphere S^{2n-1} and the author calculated $RFH_*(W_\varepsilon, \Sigma_a)$ for $n \geq 5$. He used that the chain complex $RFC_*(W_\varepsilon, \Sigma_a)$ is zero for many values of $*$ and that the boundary operator ∂^F reduces the action.
For $n = 3$ however, we encounter the problem that $RFC_*(W_\varepsilon, \Sigma_a)$ has generators for almost all $*$ so that ∂^F is not trivially zero. Using a symplectic $\mathbb{Z}_2$-symmetry on the filling V_ε we will now overcome this problem.
For convenience, assume that for $a = (2, 2, 2, a_3, \dots, a_n)$ holds that $2 < a_k \leq a_{k+1}$ for all $k \geq 3$. We find that for $0 < L < a_3$ or $0 > L > -a_3$ exactly the even L give nontrivial critical manifolds $\mathcal{N}^{L\pi/2}$, all of which are diffeomorphic to $\Sigma_{(2,2,2)}$.
Let us have a closer look at this particular Brieskorn manifold. Writing $z_k = x_k + iy_k$, we obtain from the defining equation for $\Sigma_{(2,2,2)} = V_{(2,2,2)}(0) \cap S^{2n+1}$ that

$$\begin{aligned} & 0 = z_0^2 + z_1^2 + z_2^2 && \text{and} && 1 = |z_0|^2 + |z_1|^2 + |z_2|^2 \\ \Leftrightarrow \qquad & 0 = \sum (x_k^2 - y_k^2) + 2i \sum x_k y_k && \text{and} && 1 = \sum (x_k^2 + y_k^2) \\ \Leftrightarrow \qquad & \frac{1}{2} = x_0^2 + x_1^2 + x_2^2 = y_0^2 + y_1^2 + y_2^2 && \text{and} && 0 = x_0 y_0 + x_1 y_1 + x_2 y_2. \end{aligned}$$

These equations describe the half unit tangent bundle $\frac{1}{2}S^*(\frac{1}{2}S^2)$ of the half unit sphere, which is of course naturally diffeomorphic to the true unit tangent bundle S^*S^2 of the unit sphere S^2. In Appendix A, we show that there is on S^*S^2 a perfect Morse function ψ having exactly 4 critical points such that

$$H_*(S^*S^2, \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } * = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

So if $\mathcal{N}^{L\pi/2} \cong S^*S^2$, in particular if $|L| < a_3, L \neq 0$ and L even, we get four generators for $RFC(W_\varepsilon, \Sigma_a)$. We denote them by $L\gamma_0, L\gamma_1, L\gamma_2, L\gamma_3$, where $L\gamma_j$ is the generator of $H_j(\mathcal{N}^{L\pi/2}, \mathbb{Z}_2)$. Note that these generators are exactly given by the 4 critical points of the perfect Morse function ψ. Hence, we do not need Theorem 85 here.
In order to describe $RFC_*(W_\varepsilon, \Sigma_a)$ for $*$ around 0, we calculate $\mu(L\gamma_j)$ for $|L| \leq a_3$ explicitly and give estimates for all other L. For $0 < L < a_3$, the indices are given by Proposition 103 as

$$\begin{aligned} \mu(L\gamma_j) &= 2\Big(\sum \lceil L/a_k \rceil\Big) - 2L + ind_\psi(L\gamma_j) - (n-1) \\ &= 2\Big(\underbrace{3 \cdot L/2}_{a_0, a_1, a_2} + \underbrace{n-2}_{a_3, \dots, a_n}\Big) - 2L + j - n + 1 \\ &= L + n - 3 + j. \end{aligned} \tag{65}$$

For $L \leq 0$, we estimate for any critical point $c \in \mathcal{N}^{L\pi/2}$ that

$$\begin{aligned}
\mu(c) &= 2\Big(\sum \lceil L/a_k \rceil\Big) - 2L + ind_h(c) - (n-1) \\
&\leq 2\Big(\sum L/a_k + (n+1) - n(a,L)\Big) - 2L + 2 \cdot n(a,L) - 3 - n + 1 \\
&= 2\Big(\sum 1/a_k - 1\Big) \cdot L + n \\
&\leq n.
\end{aligned} \tag{66}$$

For $L \geq a_3$, we estimate for any critical point $c \in \mathcal{N}^{L\pi/2}$ that

$$\begin{aligned}
\mu(c) &= 2\Big(\textstyle\sum_{k\geq 2} \lceil L/a_k \rceil\Big) + \underbrace{4\lceil L/2 \rceil}_{z_0, z_1} - 2L + ind_h(c) - (n-1) \\
&\geq 2\Big(\textstyle\sum_{k\geq 2} \lceil a_3/a_k \rceil\Big) + 2L - 2L + 0 - n + 1 \\
&\geq 2\Big(\underbrace{a_3/2}_{z_2} + \underbrace{n-2}_{z_3,\dots,z_n}\Big) - n + 1 \\
&= a_3 + n - 3.
\end{aligned} \tag{67}$$

For $-a_3 < L < 0$, we find by analogue calculations for the index that

$$\mu(L\gamma_j) = L - (n-1) + j. \tag{68}$$

and for any $c \in \mathcal{N}^{L\pi/2}$ the analogue estimates

$$\mu(c) \leq -a_3 - n + 4 \quad \text{if } L \leq -a_3 \qquad \text{and} \qquad \mu(c) \geq -(n-1) \quad \text{if } L \geq 0. \tag{69}$$

These calculations and estimates imply for degrees $* \in (n, a_3 + n - 3)$ respectively $* \in (-a_3 - n + 4, -n + 1)$ that the only generators in $RFC_*(W_\varepsilon, \Sigma_a)$ are the $L\gamma_j$ for even L. The distribution of these elements among the different values of $*$ is shown in the following table

$*-(n-3)$ if $L>0$ $*+(n-1)$ if $L<0$	$\vdots$	L	$L+1$	$(L+2)$	$(L+2)+1$	$(L+4)$	$\vdots$
generators	$\vdots$	$L\gamma_0$ $(L-2)\gamma_2$	$L\gamma_1$ $(L-2)\gamma_3$	$L\gamma_2$ $(L+2)\gamma_0$	$L\gamma_3$ $(L+2)\gamma_1$	$(L+4)\gamma_0$ $(L+2)\gamma_2$	$\vdots$

(70)

Theorem 109. *Let $a = (2,2,2,a_3,\dots,a_n)$ be as above. Then $RFH(W_\varepsilon, \Sigma_a)$ is independent of the filling and satisfies*

$$RFH_*(W_\varepsilon, \Sigma_a) \cong \big(\mathbb{Z}_2\big)^2$$

for $n + 2 \leq * \leq a_3 + n - 5$ or $-a_3 - n + 6 \leq * \leq -n - 1$, which is equivalent to $n + 1 \leq |*-1/2| - 1/2 \leq a_3 + n - 6$.

Proof: For Reeb orbits γ with positive period $L\pi/2 > 0$, we can estimate the Conley-Zehnder index $\mu_{CZ}(\gamma)$ by

$$\mu_{CZ}(\gamma) = 2\left(\sum \lceil L/a_k \rceil\right) - 2L \geq 2L\left(\sum 1/a_k - 1\right) \geq 2L\big(3/2 - 1\big) = L > 3 - n.$$

Theorem 65 then implies that $RFH(W_\varepsilon, \Sigma_a)$ is independent of the filling V_ε.

For the explicit calculations of homology groups, we first note that our calculations and estimates (65) - (69) and the table (70) tell us the following about the Rabinowitz-Floer chain complex. For $L > 0$ and $n+1 \leq * \leq a_3 + n - 4$ we have two cases depending on the parity of $*$

i.) $* - (n-3) = L$ is even, then $RFC_*(W_\varepsilon, \Sigma_a)$ is $\mathbb{Z}_2$-generated by

$$\begin{aligned} RFC_*(W_\varepsilon, \Sigma_a) &= \mathbb{Z}_2\langle L\gamma_0, (L-2)\gamma_2\rangle \\ RFC_{*+1}(W_\varepsilon, \Sigma_a) &= \mathbb{Z}_2\langle L\gamma_1, (L-2)\gamma_3\rangle \end{aligned}, \text{ i.e.}$$

$*$	$L+(n-3)$	$L+(n-3)+1$
gen.	$L\gamma_0$	$L\gamma_1$
	$(L-2)\gamma_2$	$(L-2)\gamma_3$

ii.) $* - (n-3) = L+1$ is odd, then $RFC_*(W_\varepsilon, \Sigma_a)$ is $\mathbb{Z}_2$-generated by

$$\begin{aligned} RFC_*(W_\varepsilon, \Sigma_a) &= \mathbb{Z}_2\langle L\gamma_1, (L-2)\gamma_3\rangle \\ RFC_{*+1}(W_\varepsilon, \Sigma_a) &= \mathbb{Z}_2\langle L\gamma_2, (L+2)\gamma_0\rangle \end{aligned}, \text{ i.e.}$$

$*$	$L+(n-3)+1$	$L+(n-3)+2$
gen.	$L\gamma_1$	$L\gamma_2$
	$(L-2)\gamma_3$	$(L+2)\gamma_0$

A similar result holds for $L < 0$ and $-a_3 - n + 5 \leq * \leq -n$ if we replace $+(n-3)$ by $-(n+1)$. As all following arguments are completely analogue for $L < 0$ or $L > 0$, we restrict ourself from now on to $L > 0$. To prove the theorem, it suffices to show that in both of the above cases the boundary operator ∂^F is zero. We will do so with the help of the following order 2 symplectic symmetry:

$$\sigma : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, \qquad (z_0, z_1, z_2, z_3, \ldots, z_n) \mapsto (z_0, z_1, -z_2, z_3, \ldots, z_n).$$

As $a_2 = 2$, it follows from Discussion 100 that σ restricts to a well-defined symplectic symmetry on V_ε. Its fixed point set V_{fix} is given by

$$V_{fix} = \big\{ z \in V_\varepsilon \,\big|\, z_2 = 0 \big\} = V_\varepsilon \cap \big(\mathbb{C}^2 \times \{0\} \times \mathbb{C}^{n-2}\big),$$

which is obviously a symplectic submanifold of V_ε. In fact, it is an exact filling of the contact manifold

$$\Sigma_{fix} = \partial V_{fix} = \big\{ z \in \Sigma_a \,\big|\, z_2 = 0 \big\} = \Sigma_a \cap \big(\mathbb{C}^2 \times \{0\} \times \mathbb{C}^{n-2}\big),$$

which is contactomorphic to the Brieskorn manifold $\Sigma_{\bar{a}}$ with $\bar{a} = (2, 2, a_3, \ldots, a_n)$.
Recall that for $0 < L < a_3$ all critical manifolds $\mathcal{N}^{L\pi/2}$ are diffeomorphic to S^*S^2 and given by

$$\mathcal{N}^{L\pi/2} = \Sigma_a \cap \big(\mathbb{C}^3 \times \{0\}\big) \cong \Sigma_{(2,2,2)}.$$

The symmetry σ restricts therefore on $\mathcal{N}^{L\pi/2} \cong S^*S^2$ to the involution

$$r : S^*S^2 \to S^*S^2, \qquad (x_0, x_1, x_2, y_0, y_1, y_2) \mapsto (x_0, x_1, -x_2, y_0, y_1, -y_2),$$

where $z_k = x_k + iy_k$. In Appendix A, we show that there exists on S^*S^2 a Morse-Smale pair (ψ, g) of a perfect Morse function and a metric which are invariant under r. Now, if the Global Transversality Theorem 42 holds, then it follows from Corollary 43 that the number of all unparametrized flow-lines with cascades between relevant generators is 0 mod 2. This would then imply that $\partial^F \equiv 0$.

It remains to show that Theorem 42 can be applied, i.e. that we can find a σ-symmetric almost complex structure J for which transversality holds for $* \in [n+1, a_3+n-4]$. Note that we can restrict ourself to this degree window, as we are only interested in $RFH_*(W_\varepsilon, \Sigma_a)$ for $* \in [n+2, a_3+n-5]$ and as the definition of the homology group $RFH_*(W_\varepsilon, \Sigma_a)$ only involves the chain groups $RFC_{*-1}(W_\varepsilon, \Sigma_a), RFC_*(W_\varepsilon, \Sigma_a)$ and $RFC_{*+1}(W_\varepsilon, \Sigma_a)$. We proceed by considering the two cases *i.)* and *ii.)* separately.

- In *i.)*, when $*-(n-3) = L$ is even, the argument goes as follows. As $RFC_*(W_\varepsilon, \Sigma_a)$ is freely generated by $L\gamma_0$ and $(L-2)\gamma_2$ and $RFC_{*+1}(W_\varepsilon, \Sigma_a)$ is freely generated by $L\gamma_1$ and $(L-2)\gamma_3$, we may write the boundary operator $\partial^F_{*+1} : RFC_{*+1} \to RFC_*$ as a 2 by 2 matrix with $\mathbb{Z}_2$-entries such that
$$\partial^F \begin{pmatrix} L\gamma_1 \\ (L-2)\gamma_3 \end{pmatrix} = \begin{pmatrix} a_{10} & a_{12} \\ a_{30} & a_{32} \end{pmatrix} \begin{pmatrix} L\gamma_0 \\ (L-2)\gamma_2 \end{pmatrix}.$$
Here, a_{ij} is the $\mathbb{Z}_2$-count of flow lines with cascades (flwc) from $L\gamma_j$ to $L\gamma_i$. For a_{10} and a_{32} these are flow lines from $L\gamma_0$ to $L\gamma_1$ resp. from $(L-2)\gamma_2$ to $(L-2)\gamma_3$. As these are between orbits of the same period, they are actually Morse flow lines and the number of such flow lines is even, as we have a perfect Morse function and thus $0 = a_{10} = a_{32}$. The coefficient a_{30} counts flwc going from $L\gamma_0$ to $(L-2)\gamma_3$. As the period can only increase along flow lines (Lemma 16), we know that there are no such flow lines and hence that $a_{30} = 0$ as well.
So it only remains to show that Theorem 42 can be applied to flow lines from $(L-2)\gamma_2$ to $L\gamma_1$ counted by a_{12}. Note that there is no closed Reeb orbit on Σ_a having a period between $(L-2)\pi/2$ and $L\pi/2$. This shows that every flow line between $(L-2)\gamma_2$ and $L\gamma_1$ has exactly 1 cascade. For Theorem 42, it suffices therefore to show that $\widehat{\mathcal{M}}\big((L-2)\gamma_2, L\gamma_1, 1\big)\big|_{V_{fix}}$ is empty for a generic choice of J on V_{fix}. For that, we calculate the indices of $(L-2)\gamma_2$ and $L\gamma_1$ in V_{fix}. As V_{fix} is V_ε with the 3^{rd}-coordinate omitted, we have by Proposition 103 that
$$\begin{aligned} \mu\big(L\gamma_1\big)\big|_{V_{fix}} &= 2\Big(\sum_{k\neq 2} \lceil L/a_k \rceil\Big) - 2L + \underbrace{0}_{ind_h(L\gamma_1)|_{V_{fix}}} -(n-2) \\ &= 2\big(\underbrace{2\cdot L/2}_{a_0,a_1} + \underbrace{n-2}_{a_3,\dots,a_n}\big) - 2L + 0 - (n-2) &&= n-2 \\ \mu\big((L-2)\gamma_2\big)\big|_{V_{fix}} &= 2\cdot\Big(\sum_{k\neq 2} \lceil (L-2)/a_k \rceil\Big) - 2(L-2) + 1 - (n-2) \\ &= 2\cdot\big(2\cdot(L-2)/2 + n-2\big) - 2(L-2) + 1 - (n-2) &&= n-1. \end{aligned}$$
Hence we obtain from Theorem 38 that for a generic J on V_{fix} we have
$$\begin{aligned} \dim \widehat{\mathcal{M}}\big((L-2)\gamma_2, L\gamma_1, 1\big)\big|_{V_{fix}} &= \mu\big(L\gamma_1\big)\big|_{V_{fix}} - \mu\big((L-2)\gamma_2\big)\big|_{V_{fix}} + (1-1) \\ &= (n-2) - (n-1) = -1, \end{aligned}$$
which shows that this space is generically empty.

- In the second case *ii.*), when $* - (n-3) = L+1$ is odd, we argue similar. We write ∂^F again as a matrix with $\mathbb{Z}_2$-coefficients such that
$$\partial^F \begin{pmatrix} L\gamma_2 \\ (L+2)\gamma_0 \end{pmatrix} = \begin{pmatrix} a_{21} & a_{23} \\ a_{01} & a_{03} \end{pmatrix} \begin{pmatrix} L\gamma_1 \\ (L-2)\gamma_3 \end{pmatrix}.$$
Here, $a_{21} = 0$ as it counts flwc from $L\gamma_1$ to $L\gamma_2$ which are Morse flow lines whose number is again even. For the remaining three coefficients, we first calculate the restricted indices as
$$\begin{aligned}
\mu\big(L\gamma_1\big)\big|_{V_{fix}} &= 2\cdot\Big(\sum_{k\neq 2}\lceil L/a_k\rceil\Big) - 2L + 0 - (n-2) \\
&= 2\cdot\big(2\cdot L/2 + n - 2\big) - 2L - (n-2) &&= n-2 \\
\mu\big(L\gamma_2\big)\big|_{V_{fix}} &= \mu\big(L\gamma_1\big)\big|_{V_{fix}} + 1 &&= n-1 \\
\mu\big((L-2)\gamma_3\big)\big|_{V_{fix}} &= 2\cdot\Big(\sum_{k\neq 2}\lceil (L-2)/a_k\rceil\Big) - 2(L-2) + 1 - (n-2) \\
&= 2\cdot\big(2\cdot(L-2)/2 + n - 2\big) - 2(L-2) + 1 - (n-2) &&= n-1 \\
\mu\big((L+2)\gamma_0\big)\big|_{V_{fix}} &= 2\cdot\Big(\sum_{k\neq 2}\lceil (L+2)/a_k\rceil\Big) - 2(L+2) + 0 - (n-2) \\
&= 2\cdot\big(2\cdot(L+2)/2 + n - 2\big) - 2(L+2) - (n-2) &&= n-2.
\end{aligned}$$
Calculating again dimensions, we find by Theorem 38 that
$$\begin{aligned}
\dim\widehat{\mathcal{M}}\big(L\gamma_1,(L+2)\gamma_0,1\big)\big|_{V_{fix}} &= (n-2)-(n-2)+0 = 0 \\
\dim\widehat{\mathcal{M}}\big((L-2)\gamma_3,L\gamma_2,1\big)\big|_{V_{fix}} &= (n-1)-(n-1)+0 = 0 \\
\dim\widehat{\mathcal{M}}\big((L-2)\gamma_3,(L+2)\gamma_0,1\big)\big|_{V_{fix}} &= (n-2)-(n-1)+0 = -1.
\end{aligned}$$
Note that $\mathbb{R}$ acts freely on $\widehat{\mathcal{M}}(\cdot,\cdot,1)$ by time shift on the non-constant cascade. This shows that all three above spaces have to be empty. As a direct consequence, we find $a_{01} = a_{23} = 0$ as there is no closed Reeb orbit whose period lies strictly between $(L-2)\pi/2$ and $L\pi/2$ resp. $L\pi/2$ and $(L+2)\pi/2$.
For a_{03} however, we have to exclude flow lines with two cascades between $(L-2)\pi/2$ and $(L+2)\pi/2$ passing through the intermediate critical manifold $\mathcal{N}^{L\pi/2}$. Here, we have to show for Theorem 42 that $\widehat{\mathcal{M}}\big((L-2)\gamma_3,\mathcal{N}^{L\pi/2}|_{V_{fix}}\big)$ and $\widehat{\mathcal{M}}\big(\mathcal{N}^{L\pi/2}|_{V_{fix}},(L+2)\gamma_0\big)$ are both empty. Noting that
$$\dim\mathcal{N}^{(L-2)\pi/2}\big|_{V_{fix}} = \dim\mathcal{N}^{L\pi/2}\big|_{V_{fix}} = \dim\mathcal{N}^{(L+2)\pi/2}\big|_{V_{fix}} = 1,$$
we calculate with the dimension formulas (17) and (18) that
$$\begin{aligned}
\dim\widehat{\mathcal{M}}\big((L-2)\gamma_3,\mathcal{N}^{L\pi/2}|_{V_{fix}}\big) &= \underbrace{2(n-2)}_{\mu_{CZ}(\mathcal{N}^{L\pi/2})} - \underbrace{2(n-2)}_{\mu_{CZ}((L-2)\gamma_3)} + \frac{1+1}{2} - 1 = 0 \\
\dim\widehat{\mathcal{M}}\big(\mathcal{N}^{L\pi/2}|_{V_{fix}},(L+2)\gamma_0\big) &= 2(n-2) - 2(n-2) + \frac{1-1}{2} + 0 = 0.
\end{aligned}$$

As $\mathbb{R}$ acts also freely on these spaces, we know again that they are empty. Hence there cannot be a flow line with two cascades between $(L-2)\gamma_3$ and $(L+2)\gamma_0$ with at least one cascade in V_{fix}. As $L\pi/2$ is the only period of a closed Reeb orbit between $(L-2)\pi/2$ and $(L+2)\pi/2$, we may hence apply Theorem 42 and find also $a_{03} = 0$. □

Remark. The result of Theorem 109 (and the ideas of its proof) together with the calculations in [23], Thm.2.10, can be used to show that the Brieskorn manifolds $\Sigma_{(2,2,2,p)}$ for p odd are all non-contactomorphic for different values of p. This improves a result shown by the author in his diploma thesis, where he could only prove that $\Sigma_{(2,2,2,p)}$ and $\Sigma_{(2,2,2,q)}$ are non-contactomorphic if $gcd(p+2, q+2) = 1$.

Corollary 110. *For the unit cotangent bundle $S^*S^2 \cong \Sigma_{(2,2,2)}$ holds that its Rabinowitz-Floer homology is independent from the filling and given by*

$$RFH_*(S^*S^2) \cong \big(\mathbb{Z}_2\big)^2 \qquad \forall * \in \mathbb{Z}.$$

Proof: The independence from the filling follows again from Theorem 65.
The critical manifolds are here all of the form $\mathcal{N}^{L\pi/2} \cong \Sigma_{(2,2,2)}$ with L even. In particular, we get for each L the four generators $L\gamma_0, L\gamma_1, L\gamma_2, L\gamma_3$ for $RFC(S^*S^2)$ as in the proof of Theorem 109. Their indices are here always

$$\mu(L\gamma_j) = 2 \cdot \Big(3 \cdot \lceil L/2 \rceil\Big) - 2L + j - (2-1) = L + j - 1.$$

Now if $* = L-1$ is odd, then $RFC_*(S^*S^2)$ is generated by $L\gamma_0$ and $(L-2)\gamma_2$, while for $* = L$ even $RFC_*(S^*S^2)$ is generated by $L\gamma_1$ and $(L-2)\gamma_3$. Thus we can use the same arguments as in the proof of Theorem 109 to show that ∂^F_* is zero for all $*$ and hence that

$$RFH_*(S^*S^2) \cong RFC_*(S^*S^2) \cong \big(\mathbb{Z}_2\big)^2.$$

□

Let us finish this section with a generalization of this corollary to the Brieskorn manifolds $\Sigma_l := \Sigma_{(2,2,2,2l)}$, $l \in \mathbb{N}$, which is due to Peter Uebele, [50]. In [20], Prop. 6.1, it is shown that the diffeomorphism type of Σ_l is given by

$$\Sigma_l \cong \begin{cases} S^2 \times S^3 & \text{if } l \equiv 0 \mod 4 \\ S^*S^3 & \text{if } l \equiv 1 \mod 4 \\ \big(S^2 \times S^3\big)\#K & \text{if } l \equiv 2 \mod 4 \\ S^*S^3\#K & \text{if } l \equiv 3 \mod 4, \end{cases}$$

where K denotes the Kervaire sphere of dimension 5 and # denotes the connected sum. As K is diffeomorphic to S^5 (see [30]) and as the cotangent bundle of S^3 is trivial, so that $S^*S^3 \cong S^2 \times S^3$, we get

Lemma 111. $\Sigma_l = \Sigma_{(2,2,2,2l)}$ *is diffeomorphic to $S^2 \times S^3$ for all $l \geq 1$.*

In [32] it is shown that all Σ_l have the same contact homology and the same holds true for their equivariant symplectic homology by [33]. Moreover, their underlying formal homotopy classes/almost contact structures coincide, as follows from [26],8.1.1 together with the fact that the first Chern classes $c_1(T\Sigma_l)$ vanishes. However, we still have:

Theorem 112 (Uebele, [50])**.** *The manifolds* $\Sigma_{(2,2,2,2l)}$, $l \geq 1$, *are non-contactomorphic.*

Proof: Our proof relies on the calculation of $RFH_*(V_\varepsilon, \Sigma_l)$. Note that it does not depend on the filling, again by Theorem 65.
For Σ_l we have two types of critical manifolds $\mathcal{N}^{L\pi/2}$, which are given by

$$\begin{aligned} \mathcal{N}^{L\pi/2} &\cong \Sigma_{(2,2,2)} \cong S^*S^2 && \text{if } l \nmid L \\ \mathcal{N}^{L\pi/2} &\cong \Sigma_{(2,2,2,2l)} \cong S^*S^3 && \text{if } l \mid L, \end{aligned}$$

where L is always even. This gives us for each L four generators of $RFC(\Sigma_l)$, which we denote again by $L\gamma_0, L\gamma_1, L\gamma_2, L\gamma_3$. Note that for $l \nmid L$, we have $L\gamma_j \in H_j(S^*S^2)$, while for $l \mid L$, we have $L\gamma_0 \in H_0(S^2 \times S^3)$, $L\gamma_1 \in H_2(S^2 \times S^3), L\gamma_2 \in H_3(S^2 \times S^3)$ and $L\gamma_3 \in H_5(S^2 \times S^3)$. The indices of these generators are given by

- $l \nmid L$

$$\mu(L\gamma_j) = 2 \cdot \Big(3 \cdot L/2 + \lceil L/2l \rceil\Big) - 2L + j - (3-1) = L + 2\lceil L/2l \rceil + j - 2$$

- $L = N(2l)$

$$\mu(L\gamma_j) = 2\left(3\frac{N2l}{2} + \frac{N2l}{2l}\right) - 2(N2l) + ind(L\gamma_j) - (3-1) = N(2l+2) + ind(L\gamma_j) - 2,$$

  where $ind(L\gamma_j) \in \{0, 2, 3, 5\}$.

For the degree $*$ around $N(2l+2) = L + 2N$, we find that the chain groups $RFC_*(\Sigma_l)$ have the following generators:

$*$	$\vdots$	$L{+}2N{-}3$	$L{+}2N{-}2$	$L{+}2N{-}1$	$L{+}2N$	$L{+}2N{+}1$	$L{+}2N{+}2$	$L{+}2N{+}3$	$L{+}2N{+}4$	$\vdots$
gen.	$\vdots$	$(L{-}2)\gamma_1$	$(L{-}2)\gamma_2$	$(L{-}2)\gamma_3$			$(L{+}2)\gamma_0$	$(L{+}2)\gamma_1$	$(L{+}2)\gamma_2$	$\vdots$
	$\vdots$	$(L{-}4)\gamma_3$	$L\gamma_0$		$L\gamma_1$	$L\gamma_2$		$L\gamma_3$	$(L{+}4)\gamma_0$	$\vdots$

Away from these values for $*$, we have only generators $L\gamma_j$ living on S^*S^2 and it follows that the situation there looks similar to the table (70). Using this explicit description of $RFC_*(\Sigma_l)$ and the arguments used in Theorem 109, we get

$$\begin{aligned} &\dim_{\mathbb{Z}_2} RFH_*(\Sigma_l) \leq 1 \quad \text{if } * = N(2l+2) + j, \text{ for any } N \in \mathbb{Z},\ j \in \{-1, 0, 1, 2\} \\ &\dim_{\mathbb{Z}_2} RFH_*(\Sigma_l) = 2 \quad \text{if } * = N(2l+2) + 5 \leq * \leq (N+1)(2l+2) - 4 \text{ for any } N \in \mathbb{Z}. \end{aligned}$$

Note that the second case is only non-empty if $l \geq 4$.

Now we can show that Σ_l and Σ_{l+k} are not contactomorphic for $l \geq 4$ and $k \geq 1$. As the Rabinowitz-Floer homology of Σ_l does not depend on the filling, it suffices to find degrees $*$, where $RFH_*(\Sigma_l) \neq RFH_*(\Sigma_{l+k})$. For $k \geq 2$, we consider $* = (2l+2)-1$ and find that

$$\dim_{\mathbb{Z}_2} RFH_*(\Sigma_l) \leq 1 \quad \text{while} \quad \dim_{\mathbb{Z}_2} RFH_*(\Sigma_{l+k}) = 2.$$

For $k = 1$, we consider $* = 2(2l+2)-1$ and find that

$$\dim_{\mathbb{Z}_2} RFH_*(\Sigma_l) \leq 1 \quad \text{while} \quad \dim_{\mathbb{Z}_2} RFH_*(\Sigma_{l+1}) = 2.$$

This shows that all Σ_l for $l \geq 4$ are non-contactomorphic. □

Remark.

- For $l = 1$, i.e. for $\Sigma_{(2,2,2,2)}$ we can use the symplectic symmetry

$$\sigma : (z_0, z_1, z_2, z_3) \mapsto (z_0, z_1, -z_2, -z_3)$$

  to show that $RFH_*(\Sigma_1) \cong \mathbb{Z}_2$ for all $* \in \mathbb{Z}$. This shows that Σ_1 is also not contactomorphic to Σ_l for any $l \geq 4$.

- For $l = 2, 3$, we can use the same arguments as above to show that they are not contactomorphic to Σ_l for any $l \geq 4$. However, our methods here do not suffice to distinguish Σ_1, Σ_2 and Σ_3. This can be achieved either by disturbing the contact structure (as did Uebele in [50]) or one would have to find a symmetric Morse function and metric on Σ_2 resp. Σ_3. Note that even though $\Sigma_2 \cong \Sigma_3 \cong S^*S^3$, the symmetry σ does here not obviously restrict to the symmetry

$$(x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3) \mapsto (x_0, x_1, -x_2, -x_3, y_0, y_1, -y_2, -y_3).$$

7.3. Exotic contact structures and fillings

In this subsection, we are going to prove our Main Theorem 114. Moreover, we show some Reeb-dynamical consequences which can be concluded from it and we give some mild generalizations.

Let us begin with the following existence result for certain contact structures on the standard sphere. It is a consequence of Theorem 109 and the handle attachment construction.

Theorem 113. *For every $n \geq 3$ and any $k \in \mathbb{Z}$ such that $k \notin [-n+1, n]$ resp. $|k - 1/2| \geq n + 1/2$ there exists on the standard sphere S^{2n-1} a contact structure ξ with filling W such that μ_{CZ} is well-defined on W, i.e. W satisfies (A) and (B), and*

$$2 \leq \dim_{\mathbb{Z}_2} RFH_k\big(W, (S^{2n-1}, \xi)\big) < \infty.$$

Proof: Consider tuples $a = (2, 2, 2, a_3, \dots, a_n)$, where $a_3, \dots, a_n$ are all odd and for all $k > 3$ holds $a_3 < a_k$ and $gcd(a_3, a_k) = 1$. It follows from Theorem 102 that the Brieskorn manifold Σ_a for such a is homeomorphic to the sphere S^{2n-1}. As the diffeomorphism types of the topological sphere S^{2n-1} form a finite group under the connected sum construction with the standard differentiable structure as neutral element (see [30]), we can find an $m \in \mathbb{N}$ such that the m-fold connected sum $\#_m\Sigma_a$ is diffeomorphic to S^{2n-1}. Let $\#_m\xi_a$ denote the resulting contact structure on S^{2n-1}. The m-fold boundary connected sum $\#_m W_\varepsilon$ of the filling W_ε of Σ_a (obtained by attaching $(m-1)$ 1-handles to m copies of W_ε) is then an exact contact filling for $(S^{2n-1}, \#_m\xi_a)$. It follows from Theorem 109 that for $a_3 \geq |k - 1/2| - n + 5 + 1/2$ holds that

$$\dim_{\mathbb{Z}_2} RFH_k(W_\varepsilon, (\Sigma_a, \xi_a)) = 2$$

and it follows from Theorem 95 for the m-fold connected sum that

$$\dim_{\mathbb{Z}_2} RFH_k\big(\#_m W_\varepsilon, (S^{2n-1}, \#_m\xi_a)\big) = \sum_{j=1}^{m} \dim_{\mathbb{Z}_2} RFH_k(W_\varepsilon, (\Sigma_a, \xi_a)) = 2m.$$

Note that $(W_\varepsilon, \Sigma_a)$ satisfies (A) and (B). Hence, it follows from Lemma 66 and 67 that $\#_m(W_\varepsilon, \Sigma_a) = (\#_m W_\varepsilon, S^{2n-1})$ also satisfies (A) and (B). □

Theorem 114 (Main Theorem).
Suppose that Σ is a differentiable manifold, $\dim \Sigma = 2n - 1 \geq 5$, which supports at least one fillable contact structure with filling for which the conditions (A) and (B) are true. Then Σ satisfies at least one of the following alternatives:

a) For every fillable contact structure ξ on Σ and any filling W of (Σ, ξ), which satisfies (A) and (B), holds true that

$$\dim_{\mathbb{Z}_2} RFH_*(W, (\Sigma, \xi)) = \infty \qquad \forall * \in \mathbb{Z} \setminus [-n+1, n].$$

b) There is (at least) one contact structure on Σ for which there exist infinitely many different fillings.

c) There exist infinitely many different fillable contact structures on Σ.

Proof:
If Σ satisfies case a) of the Main Theorem, then nothing has to be proven.
If Σ does not satisfy a), then there exists a contact structure ξ with filling W and a degree $k \in \mathbb{Z} \setminus [-n+1, n]$ such that

$$b_k^\Sigma := \dim_{\mathbb{Z}_2} RFH_k\big(W, (\Sigma, \xi)\big) < \infty.$$

By Theorem 113, we can find a fillable contact structure ξ_0 on S^{2n-1} and a filling W_0 such that

$$2 \leq \dim_{\mathbb{Z}_2} RFH_k\big(W_0, (S^{2n-1}, \xi_0)\big) =: b_k^0 < \infty.$$

By Theorem 95 we know for the connected sum of (Σ, ξ) and (S^{2n-1}, ξ_0) that

$$2m \leq \dim_{\mathbb{Z}_2} RFH_k\Big(\big(W, (\Sigma, \xi)\big)\#\big(\#_m(W_0, (S^{2n-1}, \xi_0))\big)\Big) = b_k^\Sigma + m \cdot b_k^0 \leq \infty.$$

Note that it follows from the fact that S^{2n-1} is the standard sphere that $\Sigma\#\big(\#_m S^{2n-1}\big)$ is diffeomorphic to Σ. Hence we get on Σ an infinite number of contact structures $\xi_m := \xi\#\big(\#_m\xi_0\big)$ each with an exact contact filling $W_m := W\#\big(\#_m W_0\big)$.
Now, we have two cases: If an infinite number of the contact structures ξ_m is pairwise non-contactomorphic, then Σ satisfies case c) of the Main Theorem and we are done.
If an infinite number of the contact structures ξ_m is contactomorphic to one contact structure ξ_∞, then we find that the corresponding fillings W_m of ξ_∞ cannot be equal, as the groups $RFH_k\big(W_m,(\Sigma,\xi_\infty)\big)$ are all different as their Betti-numbers $b_k^\Sigma + m\cdot b_k^0$ are all different. This implies that Σ satisfies case b) of the Main Theorem. □

Remark. One can sharpen the Main Theorem slightly by considering also the formal homotopy class $[\xi]$ of a contact structure ξ (see 1.2 for a definition). Morita showed in [40] that there are only finitely many formal homotopy classes on the standard sphere S^{4m+1} and countable infinitely many on $S^{4m+3}, m\geq 1$. His calculations also indicate that these homotopy classes on the standard sphere should form a group under connected sums with the homotopy class of the standard structure as neutral element. Hence by taking perhaps more connected sums, we can arrange in Theorem 113 that the contact structure ξ_0 on S^{4m+1} is in the standard formal homotopy class. When taking connected sums of (S^{4m+1},ξ_0) with any contact manifold (Σ,ξ), we then find that $\xi\#\xi_0$ is still in the same formal homotopy class as ξ.
On S^{4m+3}, the situation is more complicated, as the group of formal homotopy classes is infinite. However, calculations made by Uebele, [50], show that at least on S^7, S^{11} and S^{15} one can find fillable contact structures ξ with fillings W lying in the standard formal homotopy class such that

$$0<\dim_{\mathbb{Z}_2} RFH_k\big(W,(S^{4m+3},\xi)\big)<\infty.$$

I conjecture that the same holds true for any S^{4m+3}.

Let us finish this section with some dynamical consequences if Σ should satisfy *a*) or *b*) in the Main Theorem.

Theorem 115. *Let (Σ,ξ) be a compact fillable contact manifold,* $\dim\Sigma = 2n-1$*, such that for every $N\in\mathbb{N}$ there exists a filling W_N satisfying (A) and (B) and a degree $k_N\in\mathbb{Z}$ with $|k_N|\geq 3n$ such that*

$$\dim_{\mathbb{Z}_2} RFH_{k_N}(W_N,(\Sigma,\xi))>N.$$

Then it holds for any contact form α defining ξ and satisfying (MB) that its Reeb field R_α has for every $L>0$ a simple closed Reeb trajectory whose period is greater than L.

Remark.

- A closed Reeb trajectory v of period η is called simple if $v:[0,\eta)\to\Sigma$ is injective.
- Recall that according to Cieliebak and Frauenfelder, [14], appendix B, condition (MB) is satisfied for any generic contact form, i.e. for a set of second category within all contact structures defining ξ.

Proof:

1. Let α on (Σ, ξ) be an arbitrary contact form defining ξ and satisfying the Morse-Bott assumption (MB). Recall that the chain complex $RFC_k(W_N, \Sigma)$ is generated by all critical points of a Morse function h on the critical manifold
$$crit(\mathcal{A}^H) = \bigcup_{\eta \in spec(\Sigma,\alpha)} \mathcal{N}^\eta.$$
The index of such a critical point $c \in \mathcal{N}^\eta$ is given by
$$\mu(c) = \mu_{CZ}(c) + ind_h(c) - \frac{1}{2}\dim_c\big(crit(\mathcal{A}^H)\big) + \frac{1}{2}.$$
If we assume that c corresponds not to a simple trajectory but is an l-fold iteration of a shorter closed trajectory c_0, then we can estimate $\mu(c)$ with the help of the Iteration Formula for μ_{CZ} (Lemma 60) as follows
$$\begin{aligned} \mu(c) &= l\cdot\Delta(c_0) + R_c + ind_h(c) - \frac{1}{2}\dim_c\big(crit(\mathcal{A}^H)\big) + \frac{1}{2} \\ &= l\cdot\Delta(c_0) + C_c, \\ \text{where} \qquad |C_c| &\le |R_c| + \dim_c\big(crit(\mathcal{A}^H)\big) - \frac{1}{2}\dim_c\big(crit(\mathcal{A}^H)\big) + \frac{1}{2} \qquad (71) \\ &\le 2n + \frac{1}{2}(2n-1) + \frac{1}{2} \\ &= 3n. \end{aligned}$$

2. Assume that the period of every simple Reeb trajectory c_0 lies in the interval $[-L, L]$ for some $L > 0$. As α satisfies (MB), we know by Theorem 23 that there are only finitely many $\eta \in [-L, L]$ such that $\mathcal{N}^\eta \neq 0$. As the period of simple Reeb orbits is bounded by $\pm L$, we know for every $\eta \in spec(\Sigma,\alpha)$ with $|\eta| > L$ that $\mathcal{N}^\eta$ consists solely of iterated trajectories. By considering perhaps the connected components of $\mathcal{N}^\eta$ separately, we find that there exists an $l \in \mathbb{Z}$ and a $\eta_0 \in spec(\Sigma,\alpha)$ such that $\eta = l\eta_0$ and $|\eta_0| \le L$.
Note that $\mathcal{N}^\eta = \mathcal{N}^{\eta_0}$ as a submanifold of Σ, as the images of iterated trajectories coincide with the images of the underlying simple trajectories. This allows us to choose the same Morse function on $\mathcal{N}^\eta$ as on $\mathcal{N}^{\eta_0}$. Hence, we may assume for every $c \in crit(h)$, whose period is not in $[-L, L]$, that it is an iteration of a closed Reeb trajectory $c_0 \in crit(h)$, whose period lies in $[-L, L]$.

3. As Σ is compact and there are only finitely many $\eta_0 \in spec(\Sigma,\alpha)$ with $|\eta_0| \le L$, we know that there are only finitely many critical points of h whose period lies in $[-L, L]$. Let us number them $c_1, \ldots, c_m$ and consider the set of all their absolute mean indices $D := \{|\Delta(c_1)|, \ldots, |\Delta(c_m)|\}$. We define the number δ by
$$\delta := \begin{cases} 0 & \text{if } D = \{0\} \\ \min\big(D \setminus \{0\}\big) & \text{otherwise} \end{cases}.$$

If $\delta = 0$, then all mean indices $\Delta(c_i)$ are 0 and we obtain by (71) that the degree of all closed Reeb trajectories c is bounded by $\pm 3n$. Hence, $RFH_k(W_N, (\Sigma, \xi)) \neq 0$ can hold only for $|k| \leq 3n$, which contradicts the assumptions of the theorem.
If $\delta > 0$, then we can estimate for $|k| > 3n$ the number of critical points $c \in crit(h)$ with degree $\mu(c) = k$ as follows. The index of an l-fold iterate lc_j of c_j is by (71) given as

$$\mu(lc_j) = l \cdot \Delta(c_j) + C_{lc_j}.$$

As $|C_{lc_j}| \leq 3n$, we can have $\mu(lc_j) = k$ only if $l \cdot \Delta(c_j) \in [k - 3n, k + 3n]$. This is possible for at most

$$\frac{(k+3n) - (k-3n)}{\Delta(c_j)} \leq \left\lceil \frac{6n}{\delta} \right\rceil \quad \text{numbers } l.$$

Hence, there are at most $m \cdot \lceil 6n/\delta \rceil$ points $c \in crit(h)$ with $\mu(c) = k$. By assumption, we have that $\dim RFH_{k_N}(W_N, (\Sigma, \xi)) > N$. By the construction of RFH, we know that there have to be at least N different $c \in crit(h)$ generating $RFC_{k_N}(W_N, (\Sigma, \xi))$, i.e. where $\mu(c) = k_N$. However, N can be arbitrarily large, which contradicts the fact that there are at most $m \cdot \lceil 6n/\delta \rceil$ such critical points. □

Corollary 116.

- *If Σ satisfies alternative a) of the Main Theorem, then every fillable contact structure on Σ has for any generic contact form simple Reeb trajectories of arbitrary length.*
- *If Σ satisfies alternative b) but not a) of the Main Theorem, then there is at least one contact structure on Σ which has simple closed Reeb trajectories of arbitrary length for every generic contact form.*

Proof: The first statement is a direct consequence of Theorem 115. For the second statement note that we showed in the proof of the Main Theorem that if Σ satisfies b) but not a), then there exists a contact structure ξ_∞ with infinitely many fillings W_m and a degree k such that

$$\dim_{\mathbb{Z}_2} RFH_k\big(W_m, (\Sigma, \xi_\infty)\big) \geq 2m.$$

Then, the second statement is again a direct consequence of Theorem 115. □

Corollary 117. *Every Brieskorn manifold Σ_a supports at least 2 non-contactomorphic, exactly fillable contact structures.*

Proof: Note that on Σ_a the length of a simple closed Reeb trajectory is bounded from above by $\left(\prod a_k\right) \cdot \pi/2$ for the standard contact from λ_a. Therefore, it follows from Corollary 116 that Σ_a cannot satisfy alternative a) of the Main Theorem and if for Σ_a holds c), then there are infinitely many different contact structures and we are done. If Σ_a satisfies b) in the Main Theorem, then we know from Corollary 116 that it supports at least one contact structure ξ that has simple closed Reeb trajectories of arbitrary length for any contact form satisfying (MB). However, as λ_a satisfies (MB), we know that λ_a cannot be a contact form for ξ, which shows that ξ and ξ_a have to be different. □

Remark.

- In [38], Mark McLean showed that any manifold Σ, which supports at least one fillable contact structure with trivial Chern class, admits a contact structure ξ with filling W such that $\dim_{\mathbb{Z}_2} SH_*\big(W,(\Sigma,\xi)\big) = \infty$ for all $*$. Together with the long exact sequence (55) we find that the same holds true for Rabinowitz-Floer homology. This gives again Corollary 117.

- Using local Floer homology as in [38] or [27], it should be possible to sharpen Corollary 116 so that there are infinitely many closed simple Reeb trajectories for every contact form, not just the ones which satisfy (MB).

- Using the mean Euler characteristic for S^1-equivariant symplectic homology SH^{S^1}, it should be possible to distinguish all the contact structures on Brieskorn manifolds Σ_a that are obtained by our connected sum construction, provided that Σ_a satisfies any form of index positivity. This should in particular be possible if $\sum a_k > 1$. Consult [9] or [33] for the mean Euler characteristic and its behaviour under handle-attachment.

A. A perfect Morse function on S^*S^{n-1}

In this appendix, we show the existence of a Morse-Smale pair (ψ, g) of a Morse function ψ and a metric g on S^*S^{n-1} with ψ having exactly four critical points and (ψ, g) being invariant under the reflection of the last $4n-4$ coordinates:

$$r : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad (x_1, \dots x_n \ ; \ y_1, \dots, y_n) \mapsto (x_1, x_2, -x_3, \dots, -x_n \ ; \ y_1, y_2, -y_3, \dots, -y_n).$$

In the first part, we construct the Morse function ψ and calculate its critical points with their indices. In the second part, we construct g on S^*S^2 and show that (ψ, g) is Morse-Smale there. The third part finally contains the generalization of g to higher dimensions.

A.1. The Morse function ψ

This first part was (with some minor mistakes) already included in the author's diploma thesis. We repeat it here for completeness and to give a corrected version.

We consider the unit tangent bundle $S^*S^{n-1} \subset \mathbb{R}^{2n}$ of the unit sphere S^{n-1}, i.e. the set

$$S^*S^{n-1} := \left\{ z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, ||x||^2 = 1 = ||y||^2, \ \langle x, y \rangle = 0 \right\}. \tag{72}$$

The tangent space $T_z S^*S^{n-1} \subset \mathbb{R}^{2n}$ at a point $z = (x, y) \in S^*S^{n-1}$ is given by

$$T_z S^*S^{n-1} = \left\{ (\xi_x, \xi_y) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, \langle \xi_x, x \rangle = 0 = \langle \xi_y, y \rangle, \langle \xi_x, y \rangle + \langle x, \xi_y \rangle = 0 \right\}.$$

We choose $a \in \mathbb{R} \setminus \{-1, 0, 1\}$ and define the function $\psi : \mathbb{R}^{2n} \to \mathbb{R}$ as follows:

$$z_a := (x_a; y_a) = \big(\underbrace{a, 0, \dots, 0}_{x_a} \ ; \ \underbrace{0, 1, 0, \dots, 0}_{y_a} \big)$$

$$\begin{aligned} \psi(z) := \frac{1}{2} ||z - z_a||^2 \ &= \frac{1}{2} \Big(||x - x_a||^2 + ||y - y_a||^2 \Big) \\ &= \frac{1}{2} \left((x_1 - a)^2 + x_2^2 + y_1^2 + (y_2 - 1)^2 + \sum_{k=3}^{n} (x_k^2 + y_k^2) \right). \end{aligned}$$

Proposition 118. *ψ is a Morse function with four critical points, whose Morse indices are $0, n-2, n-1$ and $2n-3$. Moreover $\psi \circ r = \psi$.*

Proof: The fact that $\psi \circ r = \psi$ is obvious. The proof that ψ is a Morse function is organized in two parts:

1^{st} **claim:** ψ has four critical points.

Proof: We calculate, that the gradient of ψ on $\mathbb{R}^{2n}$ is given by

$$\nabla_z \psi = (x_1 - a, x_2, \dots, x_n \ ; \ y_1, y_2 - 1, y_3, \dots, y_n) = (x, y) - (x_a, y_a).$$

Using the theorem of extrema with constraints, we find that in a critical point (x, y) there exist real numbers $\alpha, \beta, \gamma \in \mathbb{R}$, such that with $k \geq 3$ holds

$$\begin{aligned} x_1 - a &= \alpha \cdot x_1 + \gamma \cdot y_1 & y_1 &= \beta \cdot y_1 + \gamma \cdot x_1 \\ x_2 &= \alpha \cdot x_2 + \gamma \cdot y_2 & y_2 - 1 &= \beta \cdot y_2 + \gamma \cdot x_2 \\ x_k &= \alpha \cdot x_k + \gamma \cdot y_k & y_k &= \beta \cdot y_k + \gamma \cdot x_k. \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} \mathrm{I} : (1-\alpha) \cdot x_1 &= a + \gamma \cdot y_1 & \mathrm{IV} : (1-\beta) \cdot y_1 &= \gamma \cdot x_1 \\ \mathrm{II} : (1-\alpha) \cdot x_2 &= \gamma \cdot y_2 & \mathrm{V} : (1-\beta) \cdot y_2 &= 1 + \gamma \cdot x_2 \\ \mathrm{III} : (1-\alpha) \cdot x_k &= \gamma \cdot y_k & \mathrm{VI} : (1-\beta) \cdot y_k &= \gamma \cdot x_k. \end{aligned}$$

Using III and VI, we obtain

$$(1-\alpha)(1-\beta) \cdot x_k = \gamma^2 \cdot x_k \quad \text{and} \quad (1-\alpha)(1-\beta) \cdot y_k = \gamma^2 \cdot y_k.$$

If $x_k \neq 0$ or $y_k \neq 0$ for any $k \geq 3$, we find that $(1-\alpha)(1-\beta) = \gamma^2$ and hence

$$\begin{aligned} &(1-\alpha) \cdot \mathrm{V} - \gamma \cdot \mathrm{II} &\Rightarrow\quad &(1-\alpha) = 0 \\ &\gamma \cdot \mathrm{V} - (1-\beta) \cdot \mathrm{II} &\Rightarrow\quad &\gamma = 0. \end{aligned}$$

Inserting this in I yields $a = 0$, a contradiction. Therefore, we have $x_k = y_k = 0$ for all $k \geq 3$. Hence, we are on the set

$$U_0 := S^* S^{n-1} \cap (\mathbb{R}^2 \times \mathbf{0})^2,$$

which is easily identified with $S^* S^1$. This manifold is the disjoint union of two circles and it follows that

$$y_2 = \pm x_1, \quad x_2 = \mp y_1, \quad x_1^2 + y_1^2 = 1.$$

Inserting this in II and V yields

$$\mathrm{II}^* : (1-\alpha) \cdot y_1 = -\gamma \cdot x_1, \qquad \mathrm{V}^* : (1-\beta) \cdot \pm x_1 = 1 \mp \gamma \cdot y_1.$$

Using these, we calculate that

$$\begin{aligned} &y_1 \cdot \mathrm{I} \quad + x_1 \cdot \mathrm{II}^* &\Rightarrow\quad &-\gamma = a \cdot y_1 \\ &x_1 \cdot \mathrm{IV} \pm y_1 \cdot \mathrm{V}^* &\Rightarrow\quad &-\gamma = \mp y_1 \end{aligned}.$$

Thus $a \cdot y_1 = \mp y_1$. As $a \neq \pm 1$, this implies that $y_1 = x_2 = 0$ and hence that $x_1 = \pm 1$ and $y_2 = \pm 1$. Therefore, we have the following four critical points:

$$\begin{aligned} z_+^+ &= (+1, 0, \dots, 0\,;\, 0, +1, 0, \dots, 0); & \psi(z_+^+) &= \frac{(a-1)^2}{2} \\ z_-^+ &= (+1, 0, \dots, 0\,;\, 0, -1, 0, \dots, 0); & \psi(z_-^+) &= \frac{(a-1)^2}{2} + 2 \\ z_+^- &= (-1, 0, \dots, 0\,;\, 0, +1, 0, \dots, 0); & \psi(z_+^-) &= \frac{(a+1)^2}{2} \\ z_-^- &= (-1, 0, \dots, 0\,;\, 0, -1, 0, \dots, 0); & \psi(z_-^-) &= \frac{(a+1)^2}{2} + 2. \end{aligned}$$

□

2^{nd} **claim:** All four critical points are non-degenerate.

Proof: To prove this statement, we have to calculate the Hessian of ψ at the 4 critical points. For this purpose, we need charts of S^*S^{n-1}. Recall that the inverse of the stereographic projection gives charts on S^{n-1}. These charts for the "north-pole"$= (1, 0, \dots, 0)$ and the "south-pole"$= (-1, 0, \dots, 0)$ are of the form:

$$u^{\pm} : \mathbb{R}^{n-1} \to S^{n-1}, \quad u^{\pm}(x) = \frac{1}{1 + ||x||^2} \left(\pm 1 \mp ||x||^2, 2x_1, \dots, 2x_{n-1} \right)^T .$$

Their differentials yield charts for the tangent bundle TS^{n-1}. Explicitly:

$$D_x u^{\pm} = \frac{1}{\rho(x)^2} \begin{pmatrix} \mp 4x_1 & \mp 4x_2 & \dots & \mp 4x_{n-1} \\ 2\rho(x) - 4x_1^2 & -4x_1 x_2 & \dots & -4x_1 x_{n-1} \\ -4x_2 x_1 & 2\rho(x) - 4x_2^2 & \dots & -4x_2 x_{n-1} \\ \vdots & & \ddots & \vdots \\ -4x_{n-1} x_1 & -4x_{n-1} x_2 & \dots & 2\rho(x) - 4x_{n-1}^2 \end{pmatrix},$$

where $\rho(x) := 1 + ||x||^2$. Short calculation shows $(Du^{\pm})^T (Du^{\pm}) = \frac{4}{\rho(x)^2} \cdot Id$. This implies that the following map is an affine isometry for each $x \in \mathbb{R}^{n-1}$:

$$U^{\pm}(x) : \mathbb{R}^{n-1} \to T_{u^{\pm}(x)} S^{n-1} \subset \mathbb{R}^n, \qquad U^{\pm}(x) := \frac{\rho(x)}{2} D_x u^{\pm}.$$

It follows that $U^{\pm}(x)\big(S^{n-2}\big)$ is the unit sphere in the tangent space $T_{u^{\pm}(x)} S^{n-1}$. Using the following charts given by stereographic projections:

$$v^{\pm} : \mathbb{R}^{n-2} \to S^{n-2} \subset \mathbb{R}^{n-1}, \quad v^{\pm}(y) = \frac{1}{1 + ||y||^2} \left(\pm 1 \mp ||y||^2, 2y_1, \dots, 2y_{n-2} \right)^T ,$$

we obtain four charts around $z_+^+, z_-^+, z_+^-, z_-^-$ by

$$\begin{aligned} w_{\pm}^{\pm} &:= u^{\pm} \times (U^{\pm} \cdot v^{\pm}) : \mathbb{R}^{n-1} \times \mathbb{R}^{n-2} \to S^*S^{n-1}, \text{ where} \\ w_+^+(0) &= \big(u^+(0), U^+(0) \cdot v^+(0)\big) = z_+^+ \\ w_-^+(0) &= \big(u^+(0), U^+(0) \cdot v^-(0)\big) = z_-^+ \\ w_+^-(0) &= \big(u^-(0), U^-(0) \cdot v^+(0)\big) = z_+^- \\ w_-^-(0) &= \big(u^-(0), U^-(0) \cdot v^-(0)\big) = z_-^-. \end{aligned}$$

Here, $U^{\pm}(x) \cdot v^{\pm}(y)$ denotes the matrix multiplication.

Now, we can express ψ in these charts:

$$\begin{aligned}
\psi(w_*^{\pm}(x,y)) &= \frac{1}{2}\Big(||u^{\pm}(x) - x_a||^2 + ||U^{\pm}(x)\cdot v^*(y) - y_a||^2\Big)\\
&= \frac{1}{2}\Big(||u^{\pm}(x)||^2 - 2\left\langle u^{\pm}(x), x_a\right\rangle + ||x_a||^2\\
&\qquad + ||U^{\pm}(x)\cdot v^*(y)||^2 - 2\left\langle U^{\pm}(x)\cdot v^*(y), y_a\right\rangle + ||y_a||^2\Big)\\
&= \frac{1}{2}\Big(1 - 2\left\langle u^{\pm}(x), x_a\right\rangle + a^2 + 1 - 2\left\langle v^*(y), \left(U^{\pm}(x)\right)^T \cdot y_a\right\rangle + 1\Big)\\
&= \frac{1}{2}\Big(3 + a^2 + (\pm)2a\frac{||x||^2-1}{1+||x||^2} + (*)2\frac{||y||^2-1}{1+||y||^2}\left(1 - \frac{2x_1^2}{1+||x||^2}\right)\\
&\qquad + \frac{8x_1}{1+||y||^2}\cdot\frac{\sum y_j x_{j+1}}{1+||x||^2}\Big).
\end{aligned}$$

The third equation holds, since $||u^{\pm}(x)|| = ||v^*(y)|| = 1$. Here and above, $* \in \{+,-\}$ represents the choice of signs for y. Then, some calculation shows that

$$\begin{aligned}
\frac{\partial^2(\psi\circ w_*^{\pm})}{\partial x_k\,\partial y_l}(0,0) &= \quad 0 \qquad\qquad \text{for any } k,l\\
\frac{\partial^2(\psi\circ w_*^{\pm})}{\partial y_k\,\partial y_l}(0,0) &= \begin{cases} *\,4 & \text{if } k=l\\ \ 0 & \text{otherwise}\end{cases}\\
\frac{\partial^2(\psi\circ w_*^{\pm})}{\partial x_k\,\partial x_l}(0,0) &= \begin{cases} \ 0 & \text{if } k\neq l\\ \pm 4a & \text{if } k=l\neq 1\\ \pm 4a + (*)4 & \text{if } k=l=1\end{cases}.
\end{aligned}$$

Thus we see that all four critical points $z_+^+, z_-^+, z_+^-, z_-^-$ are non-degenerate. □

If we assume that $a > 1$, we obtain that

$$ind_\psi(z_+^+) = 0, \quad ind_\psi(z_-^+) = n-2, \quad ind_\psi(z_+^-) = n-1, \quad ind_\psi(z_-^-) = 2n-3. \qquad \square$$

A.2. The metric g on S^*S^2

We remind the reader of the well-known fact that S^*S^2 is diffeomorphic to $\mathbb{R}P^3$. The latter space has a 2-covering by S^3. The corresponding covering of S^*S^2 can be explicitly obtained by restricting the following smooth map to S^3:

$$\Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^6, \qquad \Phi(s) = \Phi\begin{pmatrix}s_1\\ s_2\\ s_3\\ s_4\end{pmatrix} = \begin{pmatrix}\Phi_1(s)\\ \Phi_2(s)\end{pmatrix} := \begin{pmatrix} s_1^2+s_2^2-s_3^2-s_4^2\\ 2(s_2s_3+s_1s_4)\\ 2(s_1s_3-s_2s_4)\\ 2(s_2s_3-s_1s_4)\\ s_1^2+s_3^2-s_2^2-s_4^2\\ -2(s_1s_2+s_3s_4)\end{pmatrix}.$$

It is not difficult to see that Φ maps S^3 into S^*S^2. In fact, Φ_1 and Φ_2 restricted to S^3 yield both the Hopf-fibration and are orthogonal to each other. Moreover, $\Phi(-s) = \Phi(s)$.

Lemma 119. *$\Phi|_{S^3}$ is a differentiable 2-covering of S^*S^2.*

Proof:
1^{st} **claim:** $\Phi : S^3 \to S^*S^2$ is surjective and $\Phi^{-1}(x,y)$ contains two preimages for every point $(x,y) \in S^*S^2$.

Proof: Let $x \in S^2$ be an arbitrary point. Its preimage under the Hopf-fibration $\Phi_1^{-1}(x)$ is a circle in S^3. We parametrize this circle by the following path:

$$\gamma(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\sin\alpha & \cos\alpha & & \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & \cos\alpha & -\sin\alpha \\ & & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}.$$

Here, $s = (s_1, s_2, s_3, s_4) \in S^3$ is a point with $\Phi_1(s) = x$. The image of this circle under Φ is given by

$$\begin{aligned} \Phi(\gamma(\alpha)) &= \begin{pmatrix} \begin{pmatrix} s_1^2 + s_2^2 - s_3^2 - s_4^2 \\ 2(s_2s_3 + s_1s_4) \\ 2(s_1s_3 - s_2s_4) \end{pmatrix} \\ \cos(2\alpha) \begin{pmatrix} 2(s_2s_3 - s_1s_4) \\ s_1^2 + s_3^2 - s_2^2 - s_4^2 \\ -2(s_1s_2 + s_3s_4) \end{pmatrix} + \sin(2\alpha) \begin{pmatrix} -2(s_2s_4 + s_1s_3) \\ 2(s_1s_2 - s_3s_4) \\ s_1^2 + s_4^2 - s_2^2 - s_3^2 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \Phi_1(s) \\ \cos(2\alpha) \cdot \Phi_2(s) + \sin(2\alpha) \cdot \big(\Phi_1(s) \times \Phi_2(s)\big) \end{pmatrix}. \end{aligned}$$

Here, $\times$ denotes as usual the cross-product in $\mathbb{R}^3$. Note that while the first part of Φ maps $\gamma(\alpha)$ to $x = \Phi_1(s)$, the second part of Φ runs twice through the unit circle of vectors orthogonal to x. Thus, we see that Φ is surjective and the preimage $\Phi^{-1}(x,y)$ for $(x,y) \in S^*S^2$ contains exactly two points s and $-s$. $\square$

2^{nd} **claim:** Φ is a local diffeomorphism.

Proof: The differential of Φ is given by

$$D\Phi = 2 \cdot \begin{pmatrix} s_1 & s_2 & -s_3 & -s_4 \\ s_4 & s_3 & s_2 & s_1 \\ s_3 & -s_4 & s_1 & -s_2 \\ -s_4 & s_3 & s_2 & -s_1 \\ s_1 & -s_2 & s_3 & -s_4 \\ -s_2 & -s_1 & -s_4 & -s_3 \end{pmatrix}.$$

Recall that an orthonormal basis of T_sS^3 at a point $s = (s_1, s_2, s_3, s_4) \in S^3$ is given by

$$\begin{aligned} v_1(s) &:= (s_2, -s_1, -s_4, s_3)^T \\ v_2(s) &:= (-s_3, -s_4, s_1, s_2)^T \\ v_3(s) &:= (-s_4, s_3, -s_2, s_1)^T. \end{aligned}$$

An easy calculation shows that

$$D\Phi(v_1) = 2 \cdot \begin{pmatrix} 0 \\ \Phi_1(s) \times \Phi_2(s) \end{pmatrix}, \qquad D\Phi(v_2) = 2 \cdot \begin{pmatrix} \Phi_1(s) \times \Phi_2(s) \\ 0 \end{pmatrix}, \\ D\Phi(v_3) = 2 \cdot \begin{pmatrix} \Phi_2(s) \\ -\Phi_1(s) \end{pmatrix}. \tag{73}$$

The image of the orthonormal basis $\{v_1, v_2, v_3\}$ of T_sS^3 under $D\Phi$ is hence an orthogonal basis of $T_{\Phi(s)}S^*S^2$ and $D\Phi_{TS^3}$ is therefore pointwise an isomorphism. It follows that Φ is a local diffeomorphism. These calculations also show that the pushforward $\Phi_* g_{S^3}$ of the standard metric g_{S^3} on S^3 is not a multiple of the standard metric on S^*S^2 coming from $\mathbb{R}^{2n}$ (as $||D\Phi(v_3)||^2 = 8$, while $||D\Phi(v_1)||^2 = 4$). $\square$

It follows from claim 1 and 2 and the fact that $\Phi(-s) = \Phi(s)$ that Φ is a 2-covering of S^*S^2. Moreover, it induces a diffeomorphism between $\mathbb{R}P^3 = S^3/_\sim$ and S^*S^2. $\square$

The metric g on S^*S^2 mentioned in the introduction is the pushforward $\Phi_* g_{S^3}$. In order to see that (ψ, g) satisfies the Morse-Smale condition, we first consider on S^3 the function

$$f : \mathbb{R}^4 \supset S^3 \to \mathbb{R}, \qquad f(s) = A_1 \cdot s_1^2 + A_2 \cdot s_2^2 + A_3 \cdot s_3^2 + A_4 \cdot s_4^2,$$

where $A_i > 0$ are pairwise different positive real numbers. Note that $f(-s) = f(s)$, so that f induces a well-defined function $\Phi_* f$ on S^*S^2. Easy calculations show that for

$$A_1 = \frac{(a-1)^2}{2}, \quad A_2 = \frac{(a-1)^2}{2} + 2, \quad A_3 = \frac{(a+1)^2}{2}, \quad A_4 = \frac{(a+1)^2}{2} + 2 \tag{74}$$

the functions ψ and $\Phi_* f$ do coincide on S^*S^2.

Lemma 120. *For A_i pairwise different positive real numbers, the function f is a Morse function on S^3 having the following 8 critical point*

$$c_1^\pm = (\pm 1, 0, 0, 0), \; c_2^\pm = (0, \pm 1, 0, 0), \; c_3^\pm = (0, 0, \pm 1, 0), \; c_4^\pm = (0, 0, 0, \pm 1).$$

Moreover, f and the standard metric g_{S^3} on S^3 are Morse-Smale, i.e. the stable and unstable manifolds of the critical points intersect transversally.

Proof:

- critical points
  The gradient $\nabla^{\mathbb{R}}$ of f on $\mathbb{R}^4$ is given by $\nabla^{\mathbb{R}} f = 2 \cdot \big(A_1 s_1, A_2 s_2, A_3 s_3, A_4 s_4\big)^T$.
  The tangent space T_sS^3 at $s \in S^3$ is given by $T_sS^3 = \big\{\xi \in \mathbb{R}^4 \,\big|\, \langle \xi, s^T \rangle = 0\big\}$.
  The gradient $\nabla f := \nabla^{S^3} f$ of f on S^3 is therefore given by

$$\nabla f = 2 \begin{pmatrix} A_1 s_1 \\ A_2 s_2 \\ A_3 s_3 \\ A_4 s_4 \end{pmatrix} - \left\langle 2 \begin{pmatrix} A_1 s_1 \\ A_2 s_2 \\ A_3 s_3 \\ A_4 s_4 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \right\rangle \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} =$$

$$= 2 \begin{pmatrix} s_1 \cdot \big(A_1 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) \\ s_2 \cdot \big(A_2 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) \\ s_3 \cdot \big(A_3 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) \\ s_4 \cdot \big(A_4 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) \end{pmatrix}.$$

Assuming for example $0 < A_1 < A_2 < A_3 < A_4$, it is not difficult to see that $\nabla f = 0$ only in the 8 points $c_1^\pm, c_2^\pm, c_3^\pm, c_4^\pm$ (Hint: Use that $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$).

- non-degeneracy

  Around the critical points $c_1^\pm, c_2^\pm, c_3^\pm, c_4^\pm$ we have the following charts in S^3 coming from stereographic projection:

$$\begin{aligned} u_1^\pm &: \mathbb{R}^3 \to S^3, & u_1^\pm(x) &= \frac{1}{1+||x||^2}\big(\pm 1 \mp ||x||^2, 2x_1, 2x_2, 2x_3\big) \\ u_2^\pm &: \mathbb{R}^3 \to S^3, & u_2^\pm(x) &= \frac{1}{1+||x||^2}\big(2x_1, \pm 1 \mp ||x||^2, 2x_2, 2x_3\big) \\ u_3^\pm &: \mathbb{R}^3 \to S^3, & u_3^\pm(x) &= \frac{1}{1+||x||^2}\big(2x_1, 2x_2, \pm 1 \mp ||x||^2, 2x_3\big) \\ u_4^\pm &: \mathbb{R}^3 \to S^3, & u_4^\pm(x) &= \frac{1}{1+||x||^2}\big(2x_1, 2x_2, 2x_3, \pm 1 \mp ||x||^2\big) \end{aligned}$$

  Using these charts, easy calculations show that

$$\frac{\partial^2 f(u_i^\pm)}{\partial x_j \partial x_k}(0) = \begin{cases} 0 & \text{if } j \neq k \\ 8A_j - 8A_i & \text{if } j = k < i \\ 8A_{j+1} - 8A_i & \text{if } j = k \geq i \end{cases}.$$

  Hence, if the A_i are pairwise different, we see that all critical points are non-degenerate.

- Morse-Smale

  As above, we consider S^3 equipped with the standard metric coming from $\mathbb{R}^4$. The ordinary differential equation $\dot\gamma = \nabla f$ for a gradient trajectory γ reads as

$$\begin{aligned} \dot\gamma_1 &= 2s_1 \cdot \big(A_1 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) =: 2s_1 \cdot B_1 \\ \dot\gamma_2 &= 2s_2 \cdot \big(A_2 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) =: 2s_2 \cdot B_2 \\ \dot\gamma_3 &= 2s_3 \cdot \big(A_3 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) =: 2s_3 \cdot B_3 \\ \dot\gamma_4 &= 2s_4 \cdot \big(A_4 - (A_1 s_1^2 + A_2 s_2^2 + A_3 s_3^2 + A_4 s_4^2)\big) =: 2s_4 \cdot B_4. \end{aligned} \tag{75}$$

  Let γ be a solution of (75). As S^3 is compact without boundary, we know that γ converges asymptotically at both ends to a critical point of f. Assuming $A_4 > A_3 > A_2 > A_1 > 0$ as in (74), we find with $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$ that $B_4 \geq 0$, where $B_4 = 0$ exactly for $s = (0,0,0,\pm 1)$. So if for a time t_0 holds that $\gamma_4(t_0) \neq 0$, then $\gamma_4(t)$ strictly increases/decreases in t to $\pm 1 = sign\, \gamma_4(t_0)$ and hence $\lim_{t\to\infty} \gamma(t) = (0,0,0,\pm 1) = (0,0,0, sign\, \gamma_4(t_0))$.

If $s_4 = 0$, then $B_3 \geq 0$, where $B_3 = 0$ exactly for $s = (0, 0, \pm 1, 0)$. So if $\gamma_4(t) = 0$ for all t and if $\gamma_3(t_0) \neq 0$ for one t_0, then the same reasoning shows that $\lim\limits_{t\to\infty} \gamma(t) = (0, 0, \pm 1, 0) = (0, 0, sign\, \gamma_3(t_0), 0)$.
Analogously, we find that if $\gamma_4(t) = \gamma_3(t) = 0$ for all t and if $\gamma_2(t_0) \neq 0$ for one t_0, then $\lim\limits_{t\to\infty} = (0, \pm 1, 0, 0) = (0, sign\, \gamma_2(t_0), 0, 0)$.
In complete analogy, we show that if $\gamma_1(t_0) \neq 0$ for one t_0, then $\lim\limits_{t\to-\infty} \gamma(t) = (\pm 1, 0, 0, 0) = (sign\, \gamma_1(t_0), 0, 0, 0)$. The same, if $\gamma_1(t) = 0$ for all t, but $\gamma_2(t_0) \neq 0$, then $\lim\limits_{t\to-\infty} \gamma(t) = (0, \pm 1, 0, 0) = (0, sign\, \gamma_2(t_0), 0, 0)$ and finally if $\gamma_1(t) = \gamma_2(t) = 0$ for all t, but $\gamma_3(t_0) \neq 0$, then $\lim\limits_{t\to-\infty} \gamma(t) = (0, 0, \pm 1, 0) = (0, 0, sign\, \gamma_3(t_0), 0)$.
This allows us to read off the stable and unstable manifolds as follows

$$\begin{aligned}
W^s(c_1^\pm) &= \left\{ s \in S^3 \mid s = (\pm 1, 0, 0, 0) \right\}, \\
W^s(c_2^\pm) &= \left\{ s \in S^3 \mid s_4 = s_3 = 0, sign(s_2) = \pm 1 \right\}, \\
W^s(c_3^\pm) &= \left\{ s \in S^3 \mid s_4 = 0, sign(s_3) = \pm 1 \right\}, \\
W^s(c_4^\pm) &= \left\{ s \in S^3 \mid sign(s_4) = \pm 1 \right\}, \\
W^u(c_1^\pm) &= \left\{ s \in S^3 \mid sign(s_1) = \pm 1 \right\}, \\
W^u(c_2^\pm) &= \left\{ s \in S^3 \mid s_1 = 0, sign(s_2) = \pm 1 \right\}, \\
W^u(c_3^\pm) &= \left\{ s \in S^3 \mid s_1 = s_2 = 0, sign(s_3) = \pm 1 \right\}, \\
W^u(c_4^\pm) &= \left\{ s \in S^3 \mid s = (0, 0, 0, \pm 1) \right\}.
\end{aligned}$$

It is now obvious, that all stable and unstable manifolds intersect transversally. □

We have seen that (f, g_{S^3}) is a Morse-Smale pair on S^3 and that $\Phi : S^3 \to S^*S^2$ is a local diffeomorphism. This implies that $(\Phi_* f, \Phi_* g_{S^3})$ is also a Morse-Smale pair on S^*S^2.
To conclude this section, we note that (f, g_{S^3}) is invariant under the reflection

$$\mathfrak{r} : \mathbb{R}^4 \to \mathbb{R}^4, (s_1, s_2, s_3, s_4) \mapsto (s_1, -s_2, -s_3, s_4).$$

Note that $\mathfrak{r}$ on S^3 is conjugate via Φ to the reflection r on S^*S^2, as defined in the introduction of this appendix. It folows that $(\Phi_* f, \Phi_* g_{S^3})$ is invariant under r. Moreover, for the A_i chosen as in (74) such that $\Phi_* f = \psi$, we find that with respect to the metric $\Phi_* g_{S^3}$ there are exactly two gradient trajectories between each pair z_-^- and z_+^-, z_+^- and z_-^+ and z_-^+ and z_+^+ corresponding to the four gradient trajectories between $(0, 0, 0, \pm 1)$ and $(0, 0, \pm 1, 0)$, $(0, 0, \pm 1, 0)$ and $(\pm 1, 0, 0, 0)$ and $(\pm 1, 0, 0, 0)$ and $(0, \pm 1, 0, 0)$ respectively. These latter gradient trajectories can be read off as the 1-dimensional intersections of the stable and unstable manifolds given above.

A.3. The metric g on S^*S^{n-1}

Recall from the calculations in (73) that $D\Phi$ maps an orthonormal basis of $T_s S^3$ to an orthogonal basis of $T_{\Phi(s)} S^*S^2$, where two vectors have length 2 and one length $2\sqrt{2}$. At a point $(x, y) \in S^*S^2$, the latter vector, $D\Phi(v_3)$, is given by $2 \cdot (y, -x)^T$.

Note that we can define a global vector field X on $\mathbb{R}^{2n}$ by

$$X(x,y) := (y,-x)^T.$$

Let $\big(\mathbb{R}X\big)^\perp$ denote the orthogonal complement of $\mathbb{R}X$ with respect to the standard metric g_{std} on $\mathbb{R}^{2n}$. Then, we have for $(x,y)\neq 0$ the splitting

$$\mathbb{R}^{2n} = \mathbb{R}\oplus\big(\mathbb{R}X\big)^\perp.$$

We define a Riemannian metric g on $\mathbb{R}^{2n}\setminus\{0\}$ by requiring that

$$g(X,X)=\frac{1}{8},\quad g\big|_{(\mathbb{R}X)^\perp}=\frac{1}{4}g_{std}\big|_{(\mathbb{R}X)^\perp}\quad\text{and}\quad g(X,Y)=0\quad\forall Y\in\big(\mathbb{R}X\big)^\perp. \tag{76}$$

Then, g coincides with $\Phi_* g_{S^3}$ in the following sense: Consider for $i\geq 3$ the sets

$$\begin{aligned} S^*S^2_i &:= S^*S^{n-1}\cap(\mathbb{R}^2\times\mathbf{0}\times\underbrace{\mathbb{R}}_{i^{\text{th}}-coord.}\times\mathbf{0})^2\\ &= \big\{(x,y)\in S^*S^{n-1}\mid x_j=y_j=0 \text{ for } j\geq 3, j\neq i\big\}. \end{aligned}$$

They are easily identified with S^*S^2. Let us denote by $\Phi_i : S^3\to S^*S^2_i$ the maps given by Φ composed with this identification. Then we have that $(\Phi_i)_*(g_{S^3}) = g|_{TS^*S^2_i}$.

In the remainder of this section we show that (ψ,g) is Morse-Smale on S^*S^{n-1}. First, we calculate the gradient $\nabla^{\mathbb{R}}_g\psi$ of ψ on $\mathbb{R}^{2n}$ with respect to g. Let $\xi\in T_z\big(\mathbb{R}^{2n}\setminus\{0\}\big)$ be any tangent vector and let $\xi=\xi_X+\xi_\perp$ be its decomposition with respect to the splitting $\mathbb{R}X\oplus\big(\mathbb{R}X\big)^\perp$. Let $\nabla^{\mathbb{R}}\psi$ be the gradient of ψ with respect to g_{std} and let $\nabla^{\mathbb{R}}\psi=\nabla^{\mathbb{R}}\psi_X+\nabla^{\mathbb{R}}\psi_\perp$ be its decomposition. Then

$$\begin{aligned} d\psi(\xi) &= g_{std}(\nabla^{\mathbb{R}}\psi,\xi) = g_{std}(\nabla^{\mathbb{R}}\psi_X,\xi_X)+g_{std}(\nabla^{\mathbb{R}}\psi_\perp,\xi_\perp)\\ &\overset{(76)}{=} 8g(\nabla^{\mathbb{R}}\psi_X,\xi_X)+4g(\nabla^{\mathbb{R}}\psi_\perp,\xi_\perp)\\ &\overset{(76)}{=} 4g(2\nabla^{\mathbb{R}}\psi_X+\nabla^{\mathbb{R}}\psi_\perp,\xi_X+\xi_\perp)\\ &= 4g(\nabla^{\mathbb{R}}\psi+\nabla^{\mathbb{R}}\psi_X,\xi). \end{aligned}$$

Thus, we find that $\nabla^{\mathbb{R}}_g\psi = 4(\nabla^{\mathbb{R}}\psi+\nabla^{\mathbb{R}}\psi_X)$. To get the gradient $\nabla_g\psi$ of ψ on S^*S^{n-1}, we now have to project $\nabla^{\mathbb{R}}_g\psi$ orthogonally (with respect to g) to TS^*S^{n-1}. In other words, we calculate

$$\nabla_g\psi = \nabla^{\mathbb{R}}_g\psi - g\left(\nabla^{\mathbb{R}}_g\psi,\tbinom{x}{0}\right)\frac{\binom{x}{0}}{\big|\big|\binom{x}{0}\big|\big|_g^2} - g\left(\nabla^{\mathbb{R}}_g\psi,\tbinom{0}{y}\right)\frac{\binom{0}{y}}{\big|\big|\binom{x}{0}\big|\big|_g^2} - g\left(\nabla^{\mathbb{R}}_g\psi,\tbinom{y}{x}\right)\frac{\binom{y}{x}}{\big|\big|\binom{y}{x}\big|\big|_g^2}.$$

As $(x,0)^T$, $(0,y)^T$ and $(y,x)^T$ are all orthogonal to $X=(y,-x)^T$ for $(x,y)\in S^*S^{n-1}$, we can replace in the above equation g by $\frac{1}{4}g_{std}$. Recalling that $\nabla^{\mathbb{R}}\psi$ was given by

$$\nabla^{\mathbb{R}}\psi = (x_1-a,x_2,\ldots,x_n\ ;\ y_1,y_2-1,y_3,\ldots,y_n)^T = (x,y)^T-(x_a,y_a)^T,$$

we then calculate

$$\nabla_g\psi = 4\left(\binom{x}{y} - \binom{x_a}{y_a} + \frac{x_2 - ay_1}{2}\binom{y}{-x} - (1-ax_1)\binom{x}{0} - (1-y_2)\binom{0}{y} - \frac{-ay_1 - x_2}{2}\binom{y}{x}\right).$$

Next, we show that all solutions of $\dot{\gamma} = \nabla_g\psi$ with $\lim_{t\to\infty}\gamma(t) = z_+^-$ and $\lim_{t\to-\infty}\gamma(t) = z_-^+$ lie in the region U_0, where $x_k = y_k = 0$ for $k \geq 3$. To this purpose, consider the Lyapunov function

$$F(x,y) := (x_1-1)^2 + \sum_{i\neq 1} x_i{}^2 + (y_2-1)^2 + \sum_{i\neq 2} y_i{}^2.$$

We shall see that F increases along γ, unless the trajectory lies entirely in U_0. As $F(z_-^+) = F(z_+^-) = 4$, this proves the statement. For the derivative of F along γ, we have

$$\begin{aligned} \nabla F &= (x,y)^T - (1,0,\ldots,0\;;\;0,1,0,\ldots,0)^T \\ \frac{d}{dt}F(\gamma(t)) = dF(\dot{\gamma}(t)) &= g_{std}(\nabla F, \nabla_g\psi) \\ &= 8\cdot(a - ax_1^2 - x_2y_1 + 1 - y_2^2 - ay_1x_2) \\ &= 8a(1-x_1^2-y_1x_2) + 8(1-y_2^2-y_1x_2). \end{aligned}$$

This is positive (so that F increases along γ) if $h_1(x,y) := 1 - x_1^2 - y_1x_2 \geq 0$ and $h_2(x,y) := 1 - y_2^2 - y_1x_2 \geq 0$ and at least one of them is truly positive. By the theorem of extrema with constrains, we find that h_1 has a critical point on S^*S^{n-1} if there exist real numbers α, β, γ such that

$$\begin{array}{rrcrr} \mathrm{I}: & -2x_1 = \alpha\cdot x_1 + \gamma\cdot y_1 & & \mathrm{IV}: & -x_2 = \beta\cdot y_1 + \gamma\cdot x_1 \\ \mathrm{II}: & -y_1 = \alpha\cdot x_2 + \gamma\cdot y_2 & & \mathrm{V}: & 0 = \beta\cdot y_2 + \gamma\cdot x_2 \\ \mathrm{III}: & 0 = \alpha\cdot x_k + \gamma\cdot y_k & & \mathrm{VI}: & 0 = \beta\cdot y_k + \gamma\cdot x_k, \end{array}$$

where $k \geq 3$. From III and VI follows

$$\alpha\beta\cdot x_k = -\beta\gamma\cdot y_k = \gamma^2\cdot x_k, \qquad \alpha\beta\cdot y_k = -\alpha\gamma\cdot x_k = \gamma^2\cdot y_k.$$

Hence, whenever $x_k \neq 0$ or $y_k \neq 0$ for at least one $k \geq 3$, i.e. if we are not on U_0, we have $\alpha\beta = \gamma^2$. Then II and V yield

$$-\beta\cdot y_1 = \alpha\beta\cdot x_1 + \beta\gamma y_2 = \gamma^2 x_2 - \gamma^2 x_2 = 0.$$

So if we are not on U_0, then either $y_1 = 0$ or $(\beta = 0) \Rightarrow (\gamma = 0) \Rightarrow (x_2 = 0)$. But for these points we have $h_1 = 1 - x_1^2 \geq 0$ with equality holding for $x_1 = \pm 1$. If we are on U_0, then $x_2 = \pm y_1$ and $x_1^2 + y_1^2 = 0$. Then we also have $h_1 \geq 0$ with equality holding for $y_1 = x_2$. As $h_1 \geq 0$ holds for its critical points and S^*S^{n-1} is compact, we find that $h_1 \geq 0$ everywhere with equality for $x_1 = \pm 1$ or on U_0 with $y_1 = x_2$.
Analog calculations show that $h_2 \geq 0$ everywhere on S^*S^{n-1} with equality holding for $y_2 = \pm 1$ or on U_0 with $y_1 = x_2$. Hence $\frac{d}{dt}F(\gamma(t)) \geq 0$ with equality holding only if $\gamma(t) \in U_0$.

Recall that U_0 is diffeomorphic to S^*S^1 – the disjoint union of two circles, one of them containing z_-^+ and z_+^-, the other containing z_+^+ and z_-^-. Moreover, $U_0 \subset S^*S_i^2$ for all i. Hence, we know that there are exactly two gradient trajectories between z_-^+ and z_+^- in U_0, corresponding to the four gradient trajectories between $(\pm 1, 0, 0, 0)$ and $(0, 0, \pm 1, 0)$ on S^3 via the maps Φ_i. As there are no gradient trajectories connecting z_-^+ and z_+^- outside U_0, we know that these two are the only ones.
Therefore, we know that $W^u(z_-^+)$ and $W^s(z_+^-)$ intersect only in U_0. In order to show that (ψ, g) is Morse-Smale, it suffices to show that these spaces intersect transversally there, as all other intersections of stable/unstable manifolds are trivially transversal due to dimensions. Let γ be one of the two trajectories in U_0. Then the discussion in the previous part shows that $T_{\gamma(t)}W^u(z_-^+)$ and $T_{\gamma(t)}W^s(z_+^-)$ do span the subspaces $T_{\gamma(t)}S^*S_i^2$ for every $3 \leq i \leq n$. As these subspaces span the whole tangent space $T_{\gamma(t)}S^*S^{n-1}$, we know that $T_{\gamma(t)}W^u(z_-^+)$ and $T_{\gamma(t)}W^s(z_+^-)$ do span the whole space as well. Hence $W^u(z_-^+) \pitchfork W^s(z_+^-)$.

As a nice bonus we get the Morse-homology of S^*S^{n-1} with $\mathbb{Z}_2$-coefficients.

Corollary 121.

$$\textit{If } n \geq 3, \textit{ then:} \qquad H_k(S^*S^{n-1}, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \textit{if } k \in \{0, n-2, n-1, 2n-3\} \\ 0 & \textit{otherwise} \end{cases}.$$

$$\textit{If } n = 2, \textit{ then:} \qquad H_k(S^*S^1, \mathbb{Z}_2) = \begin{cases} (\mathbb{Z}_2)^2 & \textit{if } k \in \{0, 1\} \\ 0 & \textit{otherwise} \end{cases}.$$

B. Some properties of convolutions

If f, g are functions on $\mathbb{R}$, then their convolution $f * g$, if existent, is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y) \cdot g(y)\, dy = \int_{-\infty}^{\infty} f(y) \cdot g(x-y)\, dy = (g * f)(x).$$

We fix a smooth non-negative function $\rho : \mathbb{R} \to \mathbb{R}$ such that

$$\operatorname{supp} \rho \subset B_1(0), \quad \rho(x) = \rho(-x) \qquad \text{and} \qquad \int_{-\infty}^{\infty} \rho(x)\, dx = 1$$

and define for any $\delta > 0$ the function ρ_δ by $\quad \rho_\delta(x) := \frac{1}{\delta} \rho\left(\frac{1}{\delta} x\right)$.

For any $g \in L^p$ the convolution $\rho_\delta * g$ is well-defined and smooth, since

$$\int_{-\infty}^{\infty} \rho_\delta(x-y) \cdot g(y)\, dy = \int_{x-\delta}^{x+\delta} \rho_\delta(x-y) \cdot g(y)\, dy$$

and

$$\frac{d}{dx}\big(\rho_\delta * g\big)(x) = \int_{-\infty}^{\infty} \left(\frac{d}{dx} \rho_\delta(x-y)\right) \cdot g(y)\, dy = \Big(\big(\frac{d}{dx}\rho_\delta\big) * g\Big)(x).$$

Note that if $\operatorname{supp} g \subset [a, b]$ then $\operatorname{supp}(\rho_\delta * g) \subset [a - \delta, b + \delta]$.

Lemma 122 (cf. [19][5], 14.10.6, or [2], Lem. 2.18)**.**
For any $g \in L^p$ holds that $\rho_\delta * g$ converges in L^p to g as $\delta \to 0$.

Observe that if f is a T-periodic function, then $g * f$ is also T-periodic for any g, since

$$(f * g)(x+T) = \int_{-\infty}^{\infty} f(x+T-y) \cdot g(y)\, dy = \int_{-\infty}^{\infty} f(x-y) \cdot g(y)\, dy = (f * g)(x).$$

Lemma 123. *Let f, g be two 1-periodic functions in L^2 and let $r_\delta \in L^2$ be such that $r_\delta(x) = r_\delta(-x)$ and $\operatorname{supp} r_\delta \subset [-\delta, \delta]$ for $\delta < 1$. Then it holds that*

$$\int_0^1 (r_\delta * f) \cdot g\, dx = \int_0^1 f \cdot (r_\delta * g)\, dx.$$

Proof: As $\operatorname{supp} r_\delta \subset [-\delta, \delta]$, we have

$$(r_\delta * f)(x) = \int_{x-\delta}^{x+\delta} r_\delta(x-y) \cdot f(y) dy \qquad \text{and} \qquad (r_\delta * g)(x) = \int_{x-\delta}^{x+\delta} r_\delta(x-y) \cdot g(y) dy.$$

We calculate

$$\int_0^1 (r_\delta * f)(x) \cdot g(x) dx = \int_0^1 \int_{x-\delta}^{x+\delta} r_\delta(x-y) \cdot f(y) \cdot g(x) dy dx = (*).$$

[5]The proof in [19] is done only for $p = 1, 2$. However, we apply this lemma solely for $p = 2$ anyway.

Abbreviate $\Box := r_\delta(x-y)\cdot f(y)\cdot g(x)dxdy$. Then we get from Fubini's Theorem

$$\begin{aligned}(*) &= \int_\delta^{1-\delta}\int_{y-\delta}^{y+\delta}\Box + \int_{-\delta}^{0}\int_{0}^{y+\delta}\Box + \int_0^\delta\int_0^{y+\delta}\Box + \int_{1-\delta}^{1}\int_{y-\delta}^{1}\Box + \int_1^{1+\delta}\int_{y-\delta}^{1}\Box \\ &= \int_\delta^{1-\delta}\int_{y-\delta}^{y+\delta}\Box + \int_{1-\delta}^{1}\int_{1}^{y+\delta}\Box + \int_0^\delta\int_0^{y+\delta}\Box + \int_{1-\delta}^{1}\int_{y-\delta}^{1}\Box + \int_0^{\delta}\int_{y-\delta}^{0}\Box \\ &= \int_0^1\int_{y-\delta}^{y+\delta}\Box.\end{aligned}$$

Here, we added 1 to x and y in the 2$^{\text{nd}}$ term and subtracted 1 from x and y in the last term. Equality holds, as f and g are 1-periodic and $r_\delta(x-y) = r_\delta\big((x+1)-(y+1)\big) = r_\delta\big((x-1)-(y-1)\big)$. Using $r_\delta(x-y) = r_\delta(y-x)$, we conclude

$$\begin{aligned}(*) &= \int_0^1\int_{y-\delta}^{y+\delta} r_\delta(x-y)\cdot f(y)\cdot g(x)dxdy \\ &= \int_0^1\int_{y-\delta}^{y+\delta} f(y)\cdot r_\delta(y-x)\cdot g(x)dxdy \qquad = \int_0^1 f(y)\cdot\big(r_\delta * g\big)(y)dy.\end{aligned}$$ $\Box$

If $f:\mathbb{R}\to\mathbb{R}^n$ is an L^p-function with components $f = (f^1,\dots,f^n)$, we define the convolution $r_\delta * f$ componentwise via $\big(r_\delta * f\big)^i := r_\delta * f^i$.

Corollary 124. *Let $A = (a_{ij})$ be an $n\times n$-matrix and f, g, r_δ as in Lemma 123. Then*

$$\int_0^1 \big(r_\delta * f\big)^T A\,g\,dt = \int_0^1 f^T A\big(r_\delta * g\big)\,dt.$$

Proof: This is a consequence of Lemma 123 and linear algebra:

$$\begin{aligned}\int_0^1 \big(r_\delta * f\big)^T A\,g\,dt &= \int_0^1 \sum_{i,j}\big(r_\delta * f\big)^i\cdot a_{ij}\cdot g^j dt = \sum_{i,j} a_{ij}\cdot\int_0^1\big(r_\delta * f^i\big)\cdot g^j dt \\ &= \sum_{i,j} a_{ij}\cdot\int_0^1 f^i\cdot\big(r_\delta * g\big)^j dt = \int_0^1\sum_{i,j} f^i\cdot a_{ij}\cdot\big(r_\delta * g\big)^j dt \\ &= \int_0^1 f^T A\big(r_\delta * g\big)\,dt.\end{aligned}$$ $\Box$

Corollary 125. *If ω is a time independent 2-form and f, g and r_δ are as above, then*

$$\int_0^1 \omega(r_\delta * f, g)dt = \int_0^1 \omega(f, r_\delta * g)dt.$$

Proof: As ω is time independent, we can write $\omega(x,y) = x^T Ay$ for a time independent antisymmetric matrix A. Then apply Corollary 124. $\Box$

C. Automatic transversality

In the proof of the Local Transversality Theorem 39, we always assumed that the solution (v, η) of the Rabinowitz-Floer equation (3) is non-constant. When defining the boundary operator ∂^F, this assumption is satisfied, as each $\mathcal{A}^H$-gradient trajectory reduces the action which excludes constant solutions. However, if we consider homotopies H_s, we cannot avoid stationary trajectories.
Note that the proof of the transversality fails in this situation, as for solutions (v, η) of (3) which are constant in s, we find that $\partial_t v - \eta X_H = 0$ for all s. This forces the second equation in $(*)$ (on page 52) to be zero. As a consequence, the choice of Y has no influence on the surjectivity of $D_{J,(v,\eta)}$.

The purpose of this appendix is to show that we have always transversality along constant (v, η). For that, it suffices to show that the vertical differential $D_{J,(v,\eta)}$ of $\mathcal{F}$ is always surjective. The domain and range of $D_{J,(v,\eta)}$ are given by

$$\begin{aligned} D_{J,(v,\eta)} &: T_J\mathcal{J}^\ell \times T_{(v,\eta)}\mathcal{B} \to \mathcal{E}_{(v,\eta)}, \\ \text{where} \quad T_{(v,\eta)}\mathcal{B} &= W^{1,p}_\delta(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \oplus W^{1,p}_\delta(\mathbb{R}, \mathbb{R}) \oplus T_{(v^-,\eta^-)}C^- \oplus T_{(v^+,\eta^+)}C^+, \\ \mathcal{E}_{(v,\eta)} &= L^p_\delta(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \oplus L^p_\delta(\mathbb{R}, \mathbb{R}). \end{aligned}$$

In fact, we show that already the restriction $D_{(v,\eta)}$ of $D_{J,(v,\eta)}$ to $T_{(v,\eta)}\mathcal{B}$ is surjective. We prove this by following closely a similar proof by Salamon, [46], Lem. 2.4. We consider only the case $p = 2$, the first step in [46]. For the transition to general p see there.

To start, choose with Lemma 29 a tubular neighborhood around (v, η). This fixes a trivialization of v^*TV, so that we may write $\mathbb{R}^{2n}$ instead of v^*TV in the expressions for $T_{(v,\eta)}\mathcal{B}$ and $\mathcal{E}_{(v,\eta)}$.
Note that we have $(v^-, \eta^-) = (v^+, \eta^+)$, as (v, η) is constant in s. This implies in particular that $C^\pm = C$. However, the two spaces $T_{(v^-,\eta^-)}C$ and $T_{(v^+,\eta^+)}C$ are not the same (recall that the identification of a complement of $\ker d(ev^+) \cap \ker d(ev^-)$ with these spaces is not natural, cf. page 49). In fact, we should think of $T_{(v^\pm,\eta^\pm)}C$ as being generated by maps $\xi^\pm$ of the form

$$\xi^-(s) := (1 - \beta(s)) \cdot \xi_0 \qquad \text{and} \qquad \xi^+(s) = \beta(s) \cdot \xi_0,$$

where $\xi_0 : S^1 \to TC$ is a tangent vector to $C \subset \mathscr{L}$ and β is a fixed monotone cut-off function such that $\beta(s) = 1$ for $s \geq 0$ and $\beta(s) = 0$ for $s \leq -1$.
Now, recall that the operator $D_{(v,\eta)}$ can be written as

$$D_{(v,\eta)} = \partial_s + A,$$

where A is a self-adjoint operator on the Hilbert space $H := L^2(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ with domain $W := W^{1,2}(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$. Note that A is s-independent, as (v, η) is constant in s and that the kernel of A consists of constant maps $\mathfrak{v} : S^1 \to \mathbb{R}^{k+1} \times \{0\}$. This holds, as $\mathbb{R}^{k+1} \times \{0\}$ is identified with TC in our chosen trivialization. Hence there exists a splitting

$$H = E^+ \oplus E^- \oplus \ker A$$

into the positive, negative and zero eigenspaces of A.

Write $A^\pm := A|_{E^\pm}$ and denote by $P^\pm : H \to E^\pm$ the orthogonal projections. The operator $-A^+$ generates a strongly continuous semigroup of operators on E^+ and A^- generates a strongly semigroup of operators on E^-. Let us denote these semigroups by $s \mapsto e^{-A^+ s}$ and $s \mapsto e^{A^- s}$, where both are defined for $s \geq 0$. Let $P^0 : H \to \ker A$ denote the orthogonal projection to the kernel of A. Now define $K : \mathbb{R} \to \mathcal{L}(H)$ by

$$K(s) := \begin{cases} e^{-A^+ s} P^+ + P^0 & \text{for } s \geq 0 \\ -e^{-A^- s} P^- & \text{for } s < 0. \end{cases}$$

This function is discontinuous at $s = 0$, strongly continuous for $s \neq 0$ and satisfies

$$\left\| K(s)\big|_{E^- \oplus E^+} \right\| \leq e^{-\delta|s|}. \tag{$*$}$$

Consider the operator $Q : L^2_\delta(\mathbb{R}, H) \to \big(W^{1,2}_\delta(\mathbb{R}, H) \cap L^2_\delta(\mathbb{R}, W)\big) \oplus T_{(v^-,\eta^-)}C \oplus T_{(v^+,\eta^+)}C$, which is defined for $\mu \in L^2_\delta(\mathbb{R}, H)$ by

$$Q\mu(s) := \int_{-\infty}^{\infty} K(s-\tau)\mu(\tau) d\tau.$$

We claim that Q is a right inverse for $D_{(v,\eta)}$, thus showing that $D_{(v,\eta)}$ is surjective. To see this, note that $\xi = Q\mu = \xi^+ + \xi^- + \xi^0$ with respect to the splitting of H, where

$$\begin{aligned} \xi^+(s) &= \int_{-\infty}^{s} e^{-A^+(s-\tau)} \mu^+(\tau) d\tau \\ \xi^-(s) &= -\int_{s}^{\infty} e^{-A^-(s-\tau)} \mu^-(\tau) d\tau \\ \xi^0(s) &= \int_{-\infty}^{s} \mu^0(\tau) d\tau. \end{aligned}$$

A simple calculation shows that $\dot{\xi}^\pm + A^\pm \xi^\pm = \mu^\pm$ and $\dot{\xi}^0 + A\xi^0 = \dot{\xi}^0 = \mu^0$, as $\xi^0(s) \in \ker A$ for all s. Hence $\dot{\xi} + A\xi = \mu$. It only remains to show that ξ is actually in the right weighted space. The exponential convergence $\xi^\pm \overset{s\to\pm\infty}{\longrightarrow} 0$ follows from the exponential convergence of μ and $(*)$. Similarly, it follows that $\xi^0 \overset{s\to-\infty}{\longrightarrow} 0$ exponentially. For $s \to \infty$ however, it might happen that ξ^0 does not even converge to 0. Nevertheless, as $\mu^0 \overset{s\to\infty}{\longrightarrow} 0$ exponentially, we certainly have that $\xi^0 \to c$ exponentially for some $c \in TC$. This allows us to write

$$\xi^0(s) = \beta(s) \cdot \xi_c + (\xi^0(s) - \beta(s) \cdot \xi_c),$$

where $\xi_c : S^1 \to TC$, $t \mapsto c$ is a constant map. It follows that $\beta \cdot \xi_c \in T_{(v^+,\eta^+)}C$ and that $(\xi^0 - \beta \cdot \xi_c)$ converges at both ends exponentially to zero.
Finally note that the space $W^{1,2}_\delta(\mathbb{R}, H) \cap L^2_\delta(\mathbb{R}, W)$ agrees with $W^{1,2}_\delta(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \oplus W^{1,2}_\delta(\mathbb{R}, \mathbb{R})$.

Remark. Note that $T_{(v^-,\eta^-)}C$ lies not in the image of Q. This is due to our specific construction of Q. In general, we cannot construct a surjective Q, as $D_{(v,\eta)}$ has a kernel consisting of ξ which are constant in s with $\xi(s) \in \ker A$.

D. Conventions

Setup: (V, λ) is the completion of a Liouville domain $\tilde{V}$ with contact boundary $M = \partial\tilde{V}$ and symplectic form $\omega = d\lambda$. $(\Sigma, \alpha) \subset V$ is an exact contact hypersurface bounding a Liouville domain $W \subset V$. $H \in C^\infty(V)$ is a defining Hamiltonian for Σ, i.e. $H^{-1}(0) = \Sigma$, H constant outside a compact set and $X_H|_\Sigma = R_\alpha$.

X_H: Hamiltonian vector fields are defined by $dH = \omega(\cdot, X_H) = -\omega(X_H, \cdot)$.

J: Almost complex structures depend on $(t, n) \in S^1 \times \mathbb{R}$ with $\sup_n ||J_t(\cdot, n)||_{C^\ell} < \infty$. They are ω-compatible and t-, n-independent on $M \times [R, \infty) \subset V$ for $R \gg 0$ with

$$J|_{\xi_M} = J_0 \qquad \text{and} \qquad J\frac{\partial}{\partial r} = R_\lambda.$$

$\mathcal{A}^H$: The action functional on the free loop space $\mathscr{L}$ times $\mathbb{R}$ is defined by

$$\mathcal{A}^H(v, \eta) = \int_0^1 \Big(\lambda(\big(\dot{v}(t)\big) - \eta H\big(v(t)\big)\Big) dt.$$

$\mathcal{N}^\eta$: Denotes the set of all closed η-periodic Reeb orbits on Σ.

∇h: We consider for the Morse complex *positive* gradient flow lines $\dot{y} = \nabla h(y)$ of a Morse function h on $crit\big(\mathcal{A}^H\big)$ with respect to a Riemannian metric g_h on $crit\big(\mathcal{A}^H\big)$.

g: The Riemannian metric on $\mathscr{L} \times \mathbb{R}$ is given by

$$g\left(\begin{pmatrix}\mathfrak{v}_1\\ \hat{\eta}_1\end{pmatrix}, \begin{pmatrix}\mathfrak{v}_2\\ \hat{\eta}_2\end{pmatrix}\right) := \int_0^1 \omega\Big(\mathfrak{v}_1(t)\,,\, J_t\big(v(t), \eta\big)\mathfrak{v}_2(t)\Big) dt + \hat{\eta}_1 \cdot \hat{\eta}_2.$$

(3): The Rabinowitz-Floer equation (3) is the *positive* gradient equation $\partial_s(v, \eta) = \nabla\mathcal{A}^H(v, \eta)$. It has (20) as a counterpart for homotopies, which is not a gradient equation. Explicitly, they read as

$$\partial_s v + J_t(v, \eta)\big(\partial_t v - \eta X_H(v)\big) = 0 \quad \text{and} \quad \partial_s\eta + \int_0^1 H\big(v(s,t)\big) dt = 0, \tag{3}$$

$$\partial_s v + J_t(v, \eta)\big(\partial_t v - \eta X_{H_s}(v)\big) = 0 \quad \text{and} \quad \partial_s\eta + \int_0^1 H_s\big(v(s,t)\big) dt = 0. \tag{20}$$

(MB): The Morse-Bott assumption is

(MB) *The set $\mathcal{N}^\eta \subset \Sigma$ formed by the η-periodic Reeb orbits is a closed submanifold for each $\eta \in \mathbb{R}$ and $T_p\mathcal{N}^\eta = \ker(D_p\phi^\eta - id)$ holds for all $p \in \mathcal{N}^\eta$.*

(A)&(B): The grading assumptions are

(A) *The map $i_* : \pi_1(\Sigma) \to \pi_1(V)$ induced by the inclusion is injective.*

(B) *The integral $I_{c_1} : \pi_2(V) \to \mathbb{Z}$ of $c_1(TV)$ vanishes on spheres.*

μ: The grading of the Rabinowitz-Floer homology is defined by

$$\mu(c) = \mu_{CZ}(v) + ind_h(c) - \frac{1}{2}dim_c\mathcal{N}^\eta + \frac{1}{2}.$$

E. References

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